Point and Interval Estimation of The Inverse Weibull Distribution Parameters under Random Censoring

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Abstract The inverse weibull (IW) distribution has many applications in problems related to medical research and life testing. In this paper, point and interval estimation of the IW distribution parameters are considered in presence of random censoring. Maximum likelihood estimation (MLE) is considered for the models unknown parameters. Asymptotic and bootstrap confidence intervals are evaluated for the unknown parameters. Simulation study is carried out to see the performance of the maximum likelihood estimators (MLEs). One real data set has reanalyzed for illustrative purpose.

Keywords*: Inverse Weibull Distribution; Maximum Likelihood Estimation; Asymptotic Confidence Interval; Bootstrap Confidence Intervals; Random Censoring Scheme.*

1. Introduction

It is well known that the Weibull probability density function (PDF) can be decreasing or unimodal, and its the hazard function (HF) can be either decreasing or increasing depending on the shape parameter. Because of the flexibility of the PDF and HF, the Weibull distribution has been used quite extensively when the data indicate a monotone HF. But it cannot be used at all if the data indicate a non-monotone and unimodal HF. In many practical situations, it is often known a priori that the hazard rate cannot be monotone. It may happen that the course of a disease is such that the mortality reaches a peak after some finite period, and then declines slowly. For example, Langlands et al. (1979) have studied breast cancer data and observed that the mortality increases initially, reaches to a peak after some time and then declines slowly i.e., associated hazard rate is modified bathtub or particularly uni-modal. Such types of data can be modeled through IW distribution. In survival studies, IW distribution has been considered by many authors. Bennette (1983) analyzed the data from the Veterans administration lung cancer trial presented by Prentice (1973) and showed that the empirical failure rates for both low and high-performance status groups were unimodal in nature. It is important to analyze such data sets with the appropriate models. If the empirical studies indicate that the hazard function might be unimodal, then the IW distribution is found to be very appropriate over Weibull distribution when data indicates the non-monotone hazard rate. Erto (1989) showed that the IW distribution provides a good fit to several data given in literature, such as the times to breakdown of an insulating fluid subject to the action of a constant tension. Calabria and Publini (1990) analyzed the point maximum likelihood function of the IW parameters and reliability in complete and censored samples. Khan et al. (2008) have discussed the classical statistical properties of IW distribution.

Keller and Kamath (1982) derived this model on the basis of physical considerations on some failures of mechanical components subject to degradation phenomena. By the analysis of mechanical components of diesel engines produced by several European vehicle or engine manufacturers, Keller et al. (1985) showed that the two-parameter IW model gave the best fit to the failure data of dynamic engine components (i.e. pistons, crankshaft, main bearings, etc.) with respect to the other distributions considered (exponential and two parameter Weibull). This result seemed to be consistent with the physical derivation of the IW distribution.

Keller and Kamath (1982) introduced the IW distribution with two parameters α and β. The pdf and cdf of the IW distribution are respectively given by:

$$f(x;\alpha,\beta) = \alpha\beta^{\alpha}x^{-(\alpha+1)}\, exp\left(-\left(\frac{x}{\beta}\right)^{-\alpha}\right), \quad x > 0\,; \alpha > 0, \beta > 0, \qquad (1)$$

and

$$F(x;\alpha,\beta) = exp\left(-\left(\frac{x}{\beta}\right)^{-\alpha}\right), \quad x > 0\,; \alpha > 0, \beta > 0, \qquad (2)$$

Where α is the shape parameter and β is a scale parameter. Figure (1) illustrated the behavior of the IW distribution at α = 1 and for some various values of β.

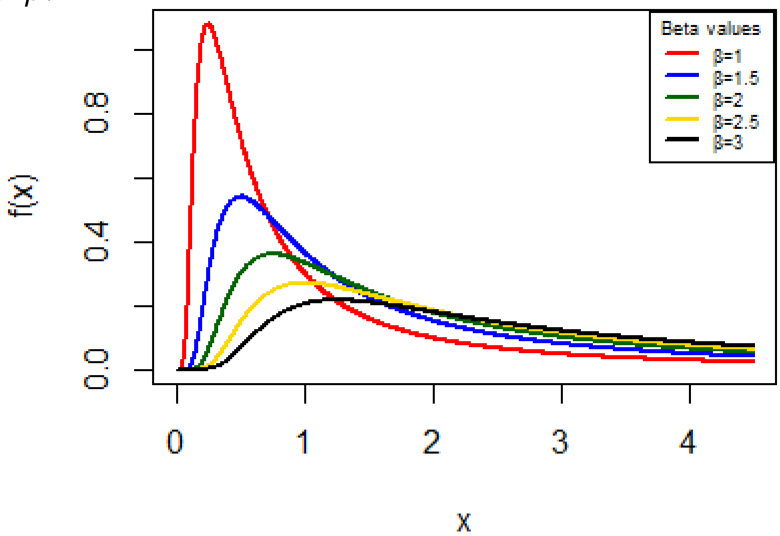

Figure 1: Density function of IW distribution for some values of β

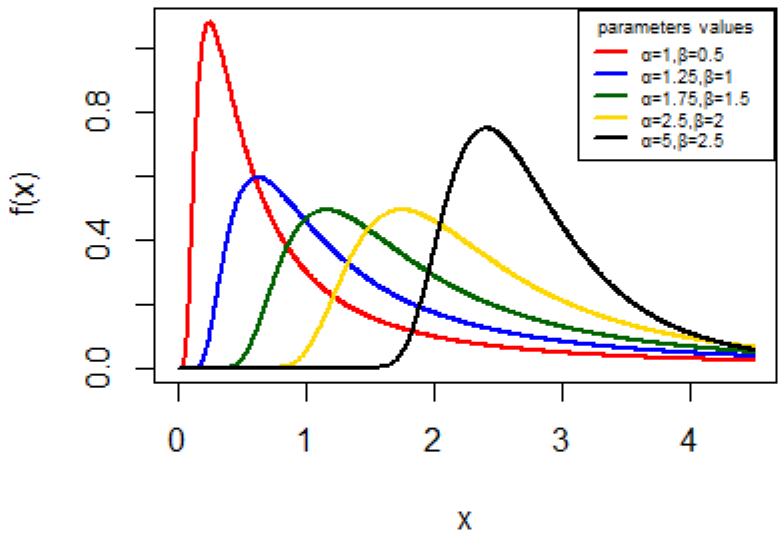

Figure 2: Density function of IW distribution for some values of α and β.

The survival function of the IW distribution is given by

$$S(x;\alpha,\beta) = 1 - exp\left(-\left(\frac{x}{\beta}\right)^{-\alpha}\right),\ \ x > 0\,;\alpha > 0, \beta > 0$$

The hazard function of the IW distribution is given by

$$h(x;\alpha,\beta) = \frac{\alpha\beta^{\alpha}x^{-(\alpha+1)}\exp\left(-\left(\frac{x}{\beta}\right)^{-\alpha}\right)}{1-\exp\left(-\left(\frac{x}{\beta}\right)^{-\alpha}\right)}\ ,\ \ x > 0\,;\alpha > 0, \beta > 0,$$

and its shape is illustrated in Figure (2) for some various values of α and β.

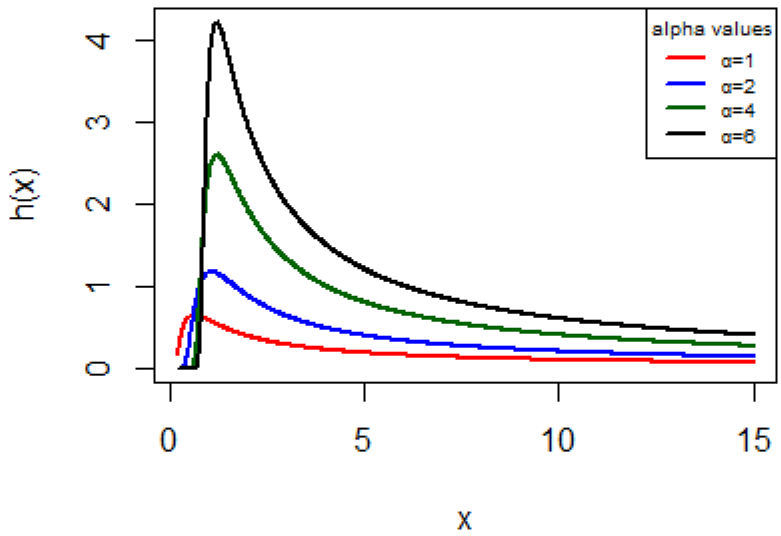

Figure 3: Hazard function of the IW distribution for different values of α when $\beta = 1$.

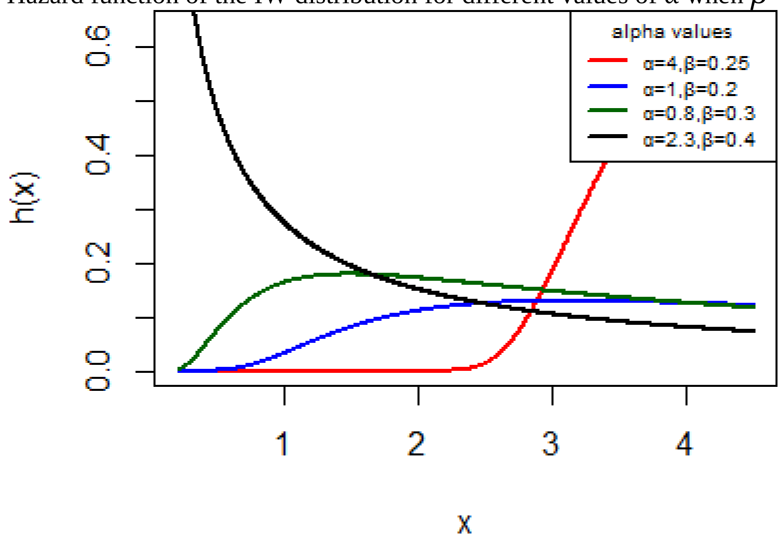

Figure 4: Hazard function of the IW distribution for different values of α and β.

In practical life testing experiment, censored data arise when the experiments including the lifetimes of test units have to be terminated before collecting complete observation. The censoring technique is common and unavoidable in practice, especially in reliability engineering, for many reasons such as time constraint and cost reduction. Various kinds of censoring have been discussed in the literature, with the most common censoring schemes being Type-I censoring, Type-II censoring and random censoring scheme.

Random censoring is a situation when an item under study is lost or removed randomly from the experiment before its failure. In other words, some subjects in the study have not experienced the event of interest at the end of the study. For example, in a clinical trial or a medical study, some patients may still be untreated and leave the course of treatment before its completion. In reliability engineering, an electrical or electronic device such as bulb on test may break before its failure.

Also, many authors have discussed various types of distributions in random censoring but they always use same distributions with different parameters for both lifetime and censoring time distributions. Rarely has considered different distributions for lifetime and censoring times in the literature. Kim (1993) considered chi-square goodness of fit tests for randomly censored data. Ghitany and Al-Awadhi (2002) analyzed in Burr Type XII distribution. Recently, Danish and Aslam (2013) discussed the Bayesian estimation for randomly censored generalized exponential distribution under asymmetric loss functions. Danish and Aslam (2014) developed the Bayesian inference for the randomly censored Weibull distribution. Krishna et al. (2015) dealt with estimation in Maxwell distribution with randomly censored data. Garg et al. (2016) considered randomly censored generalized inverted exponential distribution. Kumar and Kumar (2019) estimated the inverse weibull parameters based on random censoring data. Only Neha and Hare (2018) has considered different distributions for lifetime and dropout times, they considered clinical trials with randomly censored data having exponential healing time and Raleigh dropout times.

In this paper, we consider the lifetime units follows the IW distribution with parameters α and β and the censoring units follows the exponential distribution with scale parameters λ. The paper is organized as follows; In Section 2, a mathematical modeling is developed for randomly censored data. In Section 3, the maximum likelihood estimation method is used to obtain the point estimators of the unknown parameters. In Section 4, asymptotic and Bootstrap confidence intervals are obtained. Finally simulation results and data analysis are presented in Sections 5 and 6, respectively.

2. Model Assumption and Description

In a life testing experiment suppose n observations set under the test and their lifetimes taken as $X_1, X_2, \dots, X_n$, random variables which are identically and independently distributed (i.i.d.) with pdf $f(x;\theta)$ and cdf $F(x;\theta)$. Also, assume that $C_1, C_2, \dots, C_n$ be the random censored times of these observations. Suppose that, the pdf of $C_i's$ be $g(c;\lambda)$ and the cdf is $G(c,\lambda)$ Further, we assume $X_i's$ and $C_i's$ be mutually independent. It is noticed that, between $X_i's$ and $C_i's$ only one will actually be observed and let the actual observed time be $T_i = \min(X_i, C_i), i = 1, \dots, n$ the indicator variable δ_i is also defined as

$$\delta_i = \begin{cases} 1 & ; \quad X_i \leq C_i \\ 0 & ; \quad X_i > C_i \end{cases} \tag{3}$$

and the likelihood function under the random censoring is given by (lawless 2011).

$$L = \prod_{i=1}^{n} [f(t_i)\, R(t_i)]^{\delta_i} [g(t_i) S(t_i)]^{1-\delta_i} \tag{4}$$

Suppose that the lifetime X follows the IW distribution with unknown parameters α and β and the censoring time C independently follows exponential distribution with scale parameter λ its pdf, cdf and reliability function respectively are given by,

$$g(c;\lambda) = \lambda\, exp(-\lambda c_i) \ , \quad t, \lambda > 0,$$

$$G(c;\alpha,\beta) = 1 - ex\, p(-\lambda c_i) \, , \quad t > 0\, ; \lambda > 0,$$

and

$$R(c;\alpha,\beta) = ex\, p(-\lambda c_i) \, , \quad t > 0\, ; \lambda > 0. \tag{5}$$

3. Maximum Likelihood Estimation

In this section, we obtain the MLEs for the unknown parameters of the IW distribution based on the random censoring data. Let $(t_i, \delta_i) = (t_1, \delta_1), (t_2, \delta_2), \dots, (t_n, \delta_n)$ be a random censoring sample drowne from the model in equation (4). Then the likelihood function becomes

$$L = \prod_{i=1}^{n} \left[\alpha\beta^{\alpha} \mathrm{t}_i^{-(\alpha+1)} \exp\left(-\left(\frac{t_i}{\beta}\right)^{-\alpha}\right) exp(-\lambda t_i)\right]^{\delta_i} \left[\lambda exp(-\lambda t_i) \left(1 - \exp\left(-\left(\frac{t_i}{\beta}\right)^{-\alpha}\right)\right]^{1-\delta_i}$$

Where $\sum_{i=1}^{n} \delta_i = \mathrm{r}$ is the observed number of uncensored life time, or failures. (Lawless 2011)

Then

$$L = \left[\alpha^r \beta^{\alpha r} \prod_{i=1}^{n} t_i^{-(\alpha+1)\delta_i} \left(\exp\left(-\beta^{\alpha} \sum_{i=1}^{n} \delta_i t_i^{-\alpha}\right)\right)\left(exp\left(-\lambda \sum_{i=1}^{n} \delta_i t_i\right)\right)\right]\left[\lambda^{n-r} exp\left(-\lambda \sum_{i=1}^{n} (1 - \delta_i) t_i\right) \prod_{i=1}^{n} (1 - \exp(-\beta^{\alpha} t_i^{-\alpha}))^{1-\delta_i}\right],$$

The corresponding log likelihood function will be

$$l = rln\alpha + r\alpha ln(\beta) - (\alpha+1)\sum_{i=1}^{n} \delta_i \ln(t_i) - \beta^{\alpha} \sum_{i=1}^{n} \delta_i t_i^{-\alpha} - \lambda \sum_{i=1}^{n} \delta_i t_i + (n-r) ln\lambda - \lambda \sum_{i=1}^{n} (1-\delta_i) t_i + \sum_{i=1}^{n} (1-\delta_i) \ln(1 - \exp(-\beta^{\alpha} t_i^{-\alpha}), \quad (6)$$

Differentiating (6) with respect to α, β and λ as follows:

$$\frac{\partial l}{\partial \alpha} = \frac{r}{\alpha} + rln(\beta) - \sum_{i=1}^{n} \delta_i \ln t_i - \beta^{\alpha} \sum_{i=1}^{n} \delta_i t_i^{-\alpha} (ln\beta - \ln t_i) + \sum_{i=1}^{n} w_{1i}(\alpha, \beta)(ln\beta - \ln t_i), \quad (7)$$

$$\frac{\partial l}{\partial \beta} = \frac{\alpha r}{\beta} - \alpha\beta^{\alpha-1} \sum_{i=1}^{n} \delta_i t_i^{-\alpha} + \frac{\alpha}{\beta} \sum_{i=1}^{n} w_{1i}(\alpha, \beta), \quad (8)$$

and

$$\frac{\partial l}{\partial \lambda} = \frac{n-r}{\lambda} - \sum_{i=1}^{n} \mathrm{t}_i. \quad (9)$$

Where, $w_{1i}(\alpha, \beta) = (1 - \delta_i) \frac{exp(-\beta^{\alpha} t_i^{-\alpha}) \beta^{\alpha} t_i^{-\alpha}}{1 - exp(-\beta^{\alpha} t_i^{-\alpha})}$.

Equating the first derivations in (7), (8) and (9) to zero and solving for α, β and λ to get the MLEs $\hat{\alpha}, \hat{\beta}$ and $\hat{\lambda}$ of α, β and λ, respectively.

These equations do not yield any explicit solutions for (α and β). Therefore, these are to be solved numerically using R software as will be shown in section 5.

4. Confidence Intervals

In this section, we propose different confidence intervals. One is based on the asymptotic distribution of α, β and λ and two different bootstrap confidence intervals.

4.1 Asymptotic Confidence Intervals

The asymptotic variance-covariance matrix of the MLEs of α, β and λ can be obtained by inverting the observed information matrix $I_0^{-1}(\hat{\theta})$, and is given

$$I_0^{-1}(\hat{\theta}) = -\left.\begin{bmatrix} \frac{\partial^2 \ln L}{\partial \alpha^2} & \frac{\partial^2 \ln L}{\partial \alpha \partial \beta} & \frac{\partial^2 \ln L}{\partial \alpha \partial \lambda} \\ \frac{\partial^2 \ln L}{\partial \alpha \partial \beta} & \frac{\partial^2 \ln L}{\partial \beta^2} & \frac{\partial^2 \ln L}{\partial \beta \partial \lambda} \\ \frac{\partial^2 \ln L}{\partial \alpha \partial \lambda} & \frac{\partial^2 \ln L}{\partial \beta \partial \lambda} & \frac{\partial^2 \ln L}{\partial \lambda^2} \end{bmatrix}^{-1} \right|_{(\theta = \hat{\theta})} = \begin{bmatrix} var(\hat{\alpha}) & cov(\hat{\alpha}, \hat{\beta}) & cov(\hat{\alpha}, \hat{\lambda}) \\ cov(\hat{\alpha}, \hat{\beta}) & var(\hat{\beta}) & cov(\hat{\lambda}, \hat{\beta}) \\ cov(\hat{\alpha}, \hat{\lambda}) & cov(\hat{\lambda}, \hat{\beta}) & var(\hat{\lambda}) \end{bmatrix}.$$

Where $\hat{\theta} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda}), \theta = (\alpha, \beta, \lambda)$. The elements of the observed information matrix are given as follows:

$$\frac{\partial^2 l}{\partial \alpha^2} = -\frac{r}{\alpha^2} - \beta^\alpha \sum_{i=1}^{n} \delta_i t_i^{-\alpha} (ln\beta - lnt_i)^2 + \sum_{i=1}^{n} w_{2i}(\alpha, \beta) w_{3i}(\alpha, \beta) (ln\beta - lnt_i)^2;$$

$$\frac{\partial^2 l}{\partial \beta^2} = -\frac{\alpha r}{\beta^2} - \alpha(\alpha - 1)\beta^{\alpha-2} \sum_{i=1}^{n} \delta_i t_i^{-\alpha} + \frac{\alpha}{\beta} \sum_{i=1}^{n} w_{2i}(\alpha, \beta) w_{4i}(\alpha, \beta);$$

$$\frac{\partial^2 l}{\partial \lambda^2} = -\frac{n - r}{\lambda^2};$$

$$\frac{\partial^2 l}{\partial \alpha \partial \beta} = \frac{\partial^2 l}{\partial \beta \partial \alpha} = \frac{r}{\beta} - \alpha \beta^{\alpha-1} \sum_{i=1}^{n} \delta_i t_i \left(ln\beta - \frac{1}{\alpha} - lnt_i\right) + \sum_{i=1}^{n} w_{2i}(\alpha, \beta) w_{5i}(\alpha, \beta);$$

and

$$\frac{\partial^2 l}{\partial\beta\,\partial\lambda} = \frac{\partial^2 l}{\partial\lambda\partial\beta} = 0.$$

Where, $w_{2i}(\alpha,\beta) = (1-\delta_i)\frac{\exp(-\beta^{\alpha}t_i^{-\alpha})\beta^{\alpha}t_i^{-\alpha}}{(1-\exp(-\beta^{\alpha}t_i^{-\alpha}))^2}$,

$w_{3i}(\alpha,\beta) = [(\beta^{\alpha}t_i^{-\alpha}-1)(1-\exp(-\beta^{\alpha}t_i^{-\alpha})) - \exp(-\beta^{\alpha}t_i^{-\alpha})\,\beta^{\alpha}t_i^{-\alpha}]$,

$w_{4i}(\alpha,\beta) = \left[\left(-\alpha\beta^{\alpha-1}t_i^{-\alpha} + \frac{\alpha(\alpha-1)}{\beta}\right)(1-\exp(-\beta^{\alpha}t_i^{-\alpha})) - \alpha\beta^{\alpha-1}\exp(-\beta^{\alpha}t_i^{-\alpha})\,t_i^{-\alpha}\right]$ and

$$w_{5i}(\alpha,\beta) = \left[\left(\alpha\beta^{\alpha-1}t_i^{-\alpha}(ln\beta - lnt_i) + \frac{1}{\beta} + \frac{\alpha}{\beta}(ln\beta - lnt_i)\right)(1-\exp(-\beta^{\alpha}t_i^{-\alpha})) - \alpha\beta^{\alpha-1}t_i^{-\alpha}\exp(-\beta^{\alpha}t_i^{-\alpha})\,(ln\beta - lnt_i)\right].$$

According to particular regularity conditions, the two-sided $100(1-\gamma)\%$, $0 < \gamma < 1$, asymptotic confidence intervals for the parameters α, β and λ can be obtained as:

$$\hat{\alpha} \pm Z_{\gamma/2}\sqrt{\hat{V}_{11}}, \qquad \hat{\beta} \pm Z_{\gamma/2}\sqrt{\hat{V}_{22}}, \qquad \hat{\lambda} \pm Z_{\gamma/2}\sqrt{\hat{V}_{33}}.$$

Where $Z_{\gamma/2}$ is the upper $(\frac{\gamma}{2})$ th percentile of the standard normal distribution, $\hat{V}_{ii}$, $i = 1,2,3$ is the asymptotic variances α, β and λ, respectively.

4.2 Bootstrap Confidence Intervals

The bootstrap confidence intervals are approximate confidence interval but in general are better approximate than standard intervals. A parametric bootstrap interval provides much more information about the population value of the quantity of interest than does a point estimate. The parametric bootstrap methods are of two types:-

(i) Percentile bootstrap method (Boot-p) proposed by Efron (1993),

(ii) Bootstrap-t method (Boot-t) proposed by Hall (1988).

- Percentile Bootstrap (Boot-P) Confidence Interval

The boot-p method is rather simple and constructs confidence intervals directly from the percentiles of the bootstrap distribution of the estimated parameters. It given by the following steps:

I. A randomly censored sample is generated from the original data $T = (t_1, t_2 \dots t_n)$ and the MLE $\hat{\theta}$ of the parameter θ is computed.

II. Again, an independent randomly censored bootstrap sample $T^* = (t_1{}^*, t_2{}^* \dots t_n{}^*)$ is generated by using $\hat{\theta}$.

III. Now, compute the bootstrap MLE $\hat{\theta}^*$ of parameter θ based on T^*, as in step-1.

IV. Repeat steps 2-3, B times representing B bootstrap MLE's $\hat{\theta}^*$'s based on B different bootstrap samples, i=1, 2, … B.

V. Arrange all $\hat{\theta}^*$'s in an ascending order to obtain the bootstrap sample

i.e $\hat{\theta}^*_{(1)} \leq \hat{\theta}^*_{(2)} \leq \cdots \leq \hat{\theta}^*_{(B)}$. An approximate $100(1-\omega)\%$ boot-p confidence interval for θ is obtained by $\left(\hat{\theta}^*_{\left[\left(\frac{\omega}{2}\right)\times B\right]}, \hat{\theta}^*_{\left[\left(1-\frac{\omega}{2}\right)\times B\right]}\right)$.

Where, $\frac{\omega}{2}$ is the quantity that helps to determine the bootstrap point.

- Bootstrap-t (Boot-t) Confidence Intervals

The bootstrap-t confidence interval is given by the following steps:

I. Steps 1 and 2 of boot-p and boot-t methods are the same.

II. Compute the bootstrap-t statistic $T^* = \frac{\hat{\theta}^*{}_b - \hat{\theta}}{\sqrt{V(\hat{\theta}^*{}_b)}}$ for $\hat{\theta}^*{}_b$ where $b = 1,2,\ldots B$.

III. To obtain a set of bootstrap statistics $T^*{}_i; i = 1,2,\ldots,B$ repeat steps 2-3, B times.

IV. Let $T^*{}_{(1)} \leq T^*{}_{(2)} \leq \cdots \leq T^*{}_{(B)}$ be the ordered values of $T^*{}_i; i = 1,2,\ldots,B$.

V. Now, the approximate $100(1-\omega)\%$ boot-t confidence interval for parameter θ is obtained by

$$\left(\hat{\theta} - \hat{T}^*{}_{\left[\left(1-\frac{\omega}{2}\right)\times B\right]}\sqrt{V(\hat{\theta})}, \hat{\theta} - \hat{T}^*{}_{\left[\left(\frac{\omega}{2}\right)\times B\right]}\sqrt{V(\hat{\theta})}\right)$$

5. Simulation Study

A simulation study was carried to check the performance of the accuracy of point and interval estimates for several cases, of which estimates three parameters of IW distribution and exponential distribution (α , β and λ) for replications m=1000, for different sample size (n) as n=35,50,80,100,150 and different parameters values. All computations are obtained based on the R language.

Also, for the generation of actual observed time t from IW distribution and exponential distribution, we use the inversion method which is given by:

1. **Random number Generation for lifetime (X) from IW distribution**

$$U = F(X).$$

By substituting F(x) in (2), we get

$$u = \exp\left(-\left(\frac{x}{\beta}\right)^{-\alpha}\right).$$

Where, u is distributed as U (0, 1).

Hence,

$$x = \left(-\frac{\beta^{\alpha}}{\log(u)}\right)^{\frac{1}{\alpha}}. \qquad (10)$$

2. **Random number Generation for censoring times (C) from exponential distribution**

$$U = G(C).$$

By substituting G(C) in (4), we get

$$u = 1 - \exp(\lambda c).$$

Where, u is generated from U (0, 1).

Hence,

$$c = -\frac{\ln(1-u)}{\lambda}. \qquad (11)$$

The following steps were followed to obtain the results:

I. Specify initial values for parameters α , β and λ. as (0.9,1,0.3), (2.5,4.2,0.1) and (1.9,2,0.3)

II. Specify the sample size n. as n=35,50,80,100,150,

III. Generate m times (m=1000) of random samples of (X, C) from the model in equations (10) and (11),

IV. Determine the observed unites (t) which is the minimum of (X, C) and the indicator variable (δ) from the model (3),

V. Obtain the maximum likelihood estimates (MLEs),

VI. Obtain the mean, bias, mean squared error (MSE), asymptotic and bootstrap confidence intervals (CI's) for the unknown parameters, average interval lengths (AILs) and coverage probability (CP) for the different sample size,

VII. We assumed the lifetimes and the censoring times have the same sample size.

Discussion on simulation study

All the calculations were performed using the statistical R software. The main results of the simulation study are listed in Tables 1-3 with the following remarks.

- As expected, it is noted that the bias decreases as the sample size increases.
- The coverage probabilities for the unknown parameters are closed to 95%.
- The average length of confidence intervals decreases when sample size increases.
- Estimates obtained by maximum likelihood estimation are almost unbiased.
- Average Length of confidence intervals based on maximum likelihood estimation method increases as the parametric values increases.
- Bootstrap (t - p) confidence intervals in most cases better than the asymptotic confidence intervals.

6. Application to Real Data

In this section we analyze a real data set which consists of the survival times for 50 patients with advanced acute myelogenous leukemia reported to the international bone marrow transplant registry. The following data from Ghitany and Alawadhy (2002).

The leukemia free-survival times (in months) for the 50 patients (*) indicates the censored observations (exponential distribution), the data set is given as:

0.030, 0.493, 0.855, 1.184, 1.283, 1.480, 1.776, 2.138, 2.500, 2.763, 2.993, 3.224, 3.421, 4.178, 4.441*, 5.691, 5.855*, 6.941*, 6.941, 7.993*, 8.882, 8.882, 9.145*, 11.480, 11.513, 12.105*, 12.796, 12.993*, 13.849*, 16.612*, 17.138*, 20.066, 20.329*, 22.368*, 26.776*, 28.717*, 28.717*, 32.928*, 33.783*, 34.221*, 34.770*, 39.539*, 41.118*, 45.033*, 46.053*, 46.941*, 48.289*, 57.401*, 58.322*, 60.625*

Now, first of all, we fit the data to IW and exponential distributions. Maximum likelihood estimation methods are applied for estimating the models unknown parameters. The kolmogorov- smirnov (k-s) test is used for this purpose. With the following hypothesis:

H_0: the data come from the distribution.
H_1: the data does not come from the distribution.

Table 2: the values of goodness of fit test

Distribution	k-s	
	D-statistics	p- value
IW	0.2092	0.2907
Exponential*	0.17377	0.3664

Note: (*) indicates the censoring times distribution

We note that distance (D) value of k-s test (0.2092, 0.17377) is less than the p – value (0.2907, 0.3664). Therefore the null hypothesis does not rejected, this is lifetime data and censoring time data came from the IW Distribution and exponential distribution respectively.

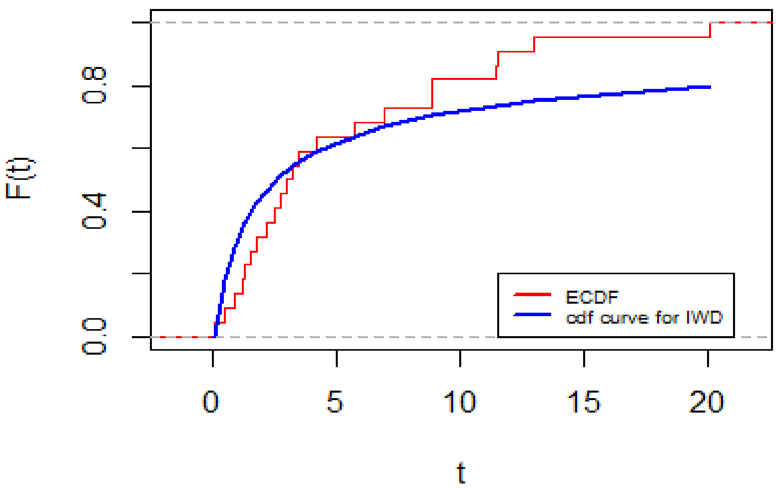

Figure 3: Empirical distribution and cdf for myelogenous data

Table 3: The Estimates of The Parameters from The Real Data Set

Parameters	MLE's	Confidence intervals		
		AILs (Asy CI)	AILs (Boot (p))	AILs (Boot (t))
$\hat{\alpha}$	0.310	0.15706 (0.2299,0.3869)	0.2568 (0.2633,0.5201)	0.18289 (0.18331,0.3662)
$\hat{\beta}$	13.349	41.92328 (4.8825,46.806)	9.15286 (7.97908,17.13194)	18.54920 (9.23039,27.77959)
$\hat{\lambda}$	0.030	0.02245 (0.0203,0.0428)	0.01121 (0.01985,0.03106)	0.01489 (0.02932,0.04421)

Note: AILs- Average interval lengths

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Table 1: Average estimated values, MSEs, bias, asymptotic CI and bootstrap (t-p) intervals of IW distribution parameters under random censoring.

N		$\alpha_0 = 0.9, \quad \beta_0 = 1, \quad \lambda_0 = 0.3$					
		Mean (MSEs)	Bias	AILs (Asymptotic CI)	AILs (Boot – P)	AILs (Boot – t)	CP
35	$\hat{\alpha}$	0.94790 (0.03288)	0.04790	0.74361 (0.65118,1.39479)	0.72342 (0.65279,1.37621)	0.70080 (0.64715,1.34795)	93.7
	$\hat{\beta}$	1.05343 (0.06577)	0.05343	1.09928 (0.69392,1.79320)	0.96978 (0.68402,1.65379)	1.18134 (0.69931,1.88065)	96.1
	$\hat{\lambda}$	0.31437 (0.00632)	0.01437	0.31191 (0.17970,0.49161)	0.31096 (0.18021,0.49118)	0.30667 (0.18567,0.49234)	94.9
50	$\hat{\alpha}$	0.92921 (0.01930)	0.02921	0.52475 (0.68349,1.20824)	0.52652 (0.70633,1.23285)	0.50844 (0.69364,1.20208)	95.6
	$\hat{\beta}$	1.02292 (0.03796)	0.02292	0.83492 (0.72141,1.55632)	0.75340 (0.70230,1.45570)	0.86426 (0.72676,1.59102)	95.9
	$\hat{\lambda}$	0.30850 (0.00454)	0.00850	0.25521 (0.19842,0.45364)	0.25198 (0.19589,0.44787)	0.25125 (0.20334,0.45459)	94.8
80	$\hat{\alpha}$	0.92216 (0.01277)	0.02216	0.42972 (0.72737,1.15709)	0.42464 (0.72043,1.14507)	0.43535 (0.73611,1.17145)	94.3
	$\hat{\beta}$	1.01803 (0.02473)	0.01803	0.62794 (0.77167,1.39961)	0.59693 (0.75502,1.35195)	0.62928 (0.77827,1.40755)	94.7
	$\hat{\lambda}$	0.30214 (0.00272)	0.00214	0.21789 (0.21398,0.43187)	0.20450 (0.21075,0.41526)	0.21248 (0.21288,0.42536)	94.3
100	$\hat{\alpha}$	0.9189 (0.0091)	0.0189	0.3645 (0.7445,1.1090)	0.3689 (0.7462,1.1151)	0.3722 (0.7433,1.1155)	97.9
	$\hat{\beta}$	1.0133 (0.0185)	0.0133	0.5500 (0.7907,1.3407)	0.5289 (0.7872,1.3161)	0.5584 (0.8011,1.3596)	97.2
	$\hat{\lambda}$	0.3038 (0.0019)	0.0038	0.1777 (0.2237,0.4013)	0.1736 (0.2262,0.3997)	0.1778 (0.2273,0.4051)	98.4
150	$\hat{\alpha}$	0.9108 (0.0061)	0.0108	0.2953 (0.7684,1.0637)	0.3100 (0.7611,1.0710)	0.3129 (0.7692,1.0822)	96.9
	$\hat{\beta}$	1.0082 (0.0119)	0.0082	0.4410 (0.8218,1.2628)	0.4316 (0.8223,1.2539)	0.4400 (0.8303,1.2703)	97.6
	$\hat{\lambda}$	0.3031 (0.0014)	0.0031	0.1444 (0.2366,0.3810)	0.1458 (0.2368,0.3826)	0.1482 (0.2366,0.3849)	97.6

Table 2: Average estimated values, MSEs, bias, asymptotic CI and bootstrap (t-p) intervals of IW distribution parameters under random censoring.

N	$\alpha_0 = 1.9,\quad \beta_0 = 2,\quad \lambda_0 = 0.3$						
		Mean (MSE)	Bias	AILs (Asymptotic CI)	AILs (Boot – P)	AILs (Boot – t)	CP
35	$\hat{\alpha}$	2.02536 (0.19022)	0.12536	1.65466 (1.31007,2.96473)	1.64869 (1.36025,3.00894)	1.61649 (1.34669,2.96318)	94.1
	$\hat{\beta}$	2.04718 (0.07417)	0.04718	1.08706 (1.64006,2.72712)	1.06026 (1.61756,2.67782)	1.08049 (1.62524,2.70573)	94.4
	$\hat{\lambda}$	0.30826 (0.00534)	0.00826	0.30573 (0.19154,0.49727)	0.28661 (0.18275,0.46936)	0.30457 (0.18739,0.49196)	93.7
50	$\hat{\alpha}$	1.97927 (0.12390)	0.07927	1.35490 (1.40493,2.75983)	1.33240 (1.39188,2.72428)	1.34836 (1.40012,2.74849)	93.3
	$\hat{\beta}$	2.02632 (0.04860)	0.02632	0.87077 (1.68010,2.55087)	0.87583 (1.66530,2.54113)	0.87448 (1.68557,2.56005)	95
	$\hat{\lambda}$	0.30704 (0.00339)	0.00704	0.22824 (0.20700,0.43524)	0.22962 (0.19909,0.42870)	0.22224 (0.21158,0.43381)	94.5
80	$\hat{\alpha}$	1.95728 (0.06260)	0.05728	0.95979 (1.50538,2.46517)	0.93992 (1.54985,2.48976)	0.95376 (1.51810,2.47187)	95.8
	$\hat{\beta}$	2.01650 (0.02662)	0.01650	0.65128 (1.74052,2.39181)	0.65199 (1.73098,2.38297)	0.66205 (1.73716,2.39921)	94.9
	$\hat{\lambda}$	0.29947 (0.00206)	0.00053	0.17707 (0.21965,0.39672)	0.17585 (0.21839,0.39424)	0.17675 (0.22395,0.40070)	95.1
100	$\hat{\alpha}$	1.9471 (0.0532)	0.0471	0.8545 (1.5419,2.3964)	0.8842 (1.5548,2.4389)	0.8773 (1.5434,2.4206)	98
	$\hat{\beta}$	2.0139 (0.0213)	0.0139	0.5774 (1.7645,2.3419)	0.5583 (1.7698,2.3281)	0.5757 (1.7600,2.3357)	97.9
	$\hat{\lambda}$	0.3009 (0.0015)	0.0009	0.1588 (0.2284,0.3872)	0.1507 (0.2309,0.3816)	0.1505 (0.2338,0.3842)	98.2
150		1.9268 (0.0313)	0.0268	0.6901 (1.5961,2.2861)	0.6875 (1.6034,2.2909)	0.6964 (1.6101,2.3065)	97.9
		2.0090 (0.0147)	0.0090	0.4669 (1.8014,2.2683)	0.4651 (1.7989,2.2640)	0.4691 (1.8089,2.2780)	97.1
		0.3020 (0.0012)	0.0020	0.1299 (0.2417,0.3716)	0.1309 (0.2414,0.3723)	0.1347 (0.2409,0.3756)	97.4

Table 3: Average estimated values, MSEs, bias, asymptotic CI and bootstrap (t-p) intervals of IW distribution parameters under random censoring.

$\alpha_0 = 2.5, \quad \beta_0 = 4.2, \quad \lambda_0 = 0.1$

N		Mean (MSE)	Bias	AILs (Asymptotic CI)	AILs (Boot – P)	AILs (Boot – t)	CP
35	$\hat{\alpha}$	2.65510 (0.26750)	0.15510	1.78608 (1.83826,3.62434)	1.93140 (1.87544,3.80684)	1.94488 (1.81088,3.75576)	94.1
	$\hat{\beta}$	4.25094 (0.13266)	0.05094	1.49657 (3.62421,5.12077)	1.38348 (3.63183,5.01531)	1.49952 (3.64612,5.14564)	96
	$\hat{\lambda}$	0.10210 (0.00074)	0.00210	0.10373 (0.05896,0.16269)	0.10282 (0.05670,0.15952)	0.10980 (0.05997,0.16977)	94.2
50	$\hat{\alpha}$	2.61835 (0.16593)	0.11835	1.46457 (1.93702,3.40159)	1.55639 (1.94677,3.50317)	1.55573 (1.94150,3.49723)	94.1
	$\hat{\beta}$	4.22636 (0.10778)	0.02636	1.21773 (3.69861,4.91634)	1.25540 (3.68101,4.93641)	1.30818 (3.67026,4.97843)	94.4
	$\hat{\lambda}$	0.10170 (0.00050)	0.00170	0.08660 (0.06452,0.15113)	0.08712 (0.06278,0.14990)	0.09018 (0.06441,0.15459)	94.6
80	$\hat{\alpha}$	2.56399 (0.09385)	0.06399	1.13037 (2.02922,3.15959)	1.21068 (2.01232,3.22300)	1.20999 (2.03665,3.24664)	93.3
	$\hat{\beta}$	4.22032 (0.05853)	0.02032	0.95190 (3.79430,4.74620)	0.95416 (3.79915,4.75331)	0.97335 (3.78866,4.76201)	95
	$\hat{\lambda}$	0.10126 (0.00029)	0.00126	0.06805 (0.07104,0.13909)	0.06670 (0.06944,0.13614)	0.06964 (0.07216,0.14179)	95
100	$\hat{\alpha}$	2.5446 (0.0744)	0.0446	1.0002 (2.0683,3.0685)	1.0949 (2.0670,3.1620)	1.0698 (2.0476,3.1175)	97.5
	$\hat{\beta}$	4.2255 (0.0495)	0.0255	0.8503 (3.8399,4.6902)	0.8631 (3.8307,4.6938)	0.9292 (3.8201,4.7493)	97.3
	$\hat{\lambda}$	0.1005 (0.0003)	0.0005	0.0605 (0.0733,0.1338)	0.0636 (0.0721,0.1357)	0.0660 (0.0715,0.1374)	96.6
150	$\hat{\alpha}$	2.5343 (0.0444)	0.0343	0.8107 (2.1445,2.9552)	0.8008 (2.1650,2.9659)	0.8032 (2.1511,2.9543)	98.1
	$\hat{\beta}$	4.2043 (0.0323)	0.0043	0.6843 (3.8879,4.5722)	0.6980 (3.8823,4.5803)	0.6987 (3.8873,4.5860)	97.1
	$\hat{\lambda}$	0.1007 (0.0002)	0.0007	0.0496 (0.0780,0.1275)	0.0498 (0.0771,0.1269)	0.0508 (0.0782,0.1290)	96.7