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Unit Root Test of Bounded AR (2) without Constant Model in Case Dependent Errors

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Unit Root Test of Bounded AR (2) without Constant Model in Case Dependent Errors

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Abstract

In this paper, the test of unit root for bounded AR (2) model without constant and dependent errors has been derived. Asymptotic distributions of OLS estimators and $t-type$ statistics under different tests of hypotheses have been derived. A simulation study has been established to compare between different tests of the unit root. Mean squared error (MSE) and Thiel's inequality coefficient (Thiel's U) have been considered as criteria of comparison.

Keywords: Bounded AR (2) model without constant, dependent errors**,** asymptotic distributions, OLS estimators, tests of hypothesis, the $t-type$ statistics, Mean squared error, Thiel's inequality coefficient and power of the test.

1. Introduction

Many unit root tests have been developed for testing the null hypothesis of a unit root against the alternative of stationarity, the tests for unit roots in AR (1) processes were first proposed and investigated by Dickey and Fuller (1979, 1981) but these unit root tests are proposed to unbounded time series in case of independent error terms.

Cavaliere (2000) tested the presence of unknown boundaries which constrain the sample path to lie within a closed interval that is in the framework of integrated processes of AR (1) model with a unit root or random walk model (with and without linear trend) and in

(2002) he introduced the logged nominal exchange rates $\{y_t\}$ that change in time accordingly to a first-order integrated process, $I(1)$ within the framework of non-managed flexible exchange rates. In (2005), Cavaliere developed an asymptotic theory for integrated and near-integrated time series whose range is constrained in some ways. Such a framework arises when integration and cointegration analysis are applied to persistent series which are bounded either by construction or because they are subject to control.

Cavaliere and Xu (2011) defined bounded process as time series x_t with (fixed) bounds at $\underline{b}, \bar{b}; \underline{b} < \bar{b}$, is a stochastic process satisfying $x_t \in [\underline{b}, \bar{b}]$ for all *t*. Carrion and Gadea (2013) showed that the use of generalized least squares (GLS) detrending procedures leads to important empirical power gains compared to ordinary least squares (OLS) detrending method when testing the null hypothesis of unit root for bounded processes. In (2015), they discussed the unit root testing when the range of the time series is bounded considering the presence of multiple structural breaks. But they all concentrated on the model of bounded AR (1) with constant or without constant under various assumptions for the error terms, and in this paper the concentration will be on the bounded AR (2) without constant model in case of dependent errors.

2. Test of Unit Root for Bounded AR (2) Model without Constant in Case of Dependent Errors

The bounded second order autoregressive AR (2) model without constant takes the form:

$$y_t = \rho_1 y_{t-1} + \rho_2 y_{t-2} + u_t, \qquad t = 1, \ldots, T, \qquad (1)$$

where y_t is bounded time series with fixed bounds with lower bound at $\underline{b}$ and upper bound at $\bar{b}$, $y_t \in [\underline{b}, \bar{b}]$, and $\underline{b} = \underline{c}\, T^{1/2} [1 - \phi_1]^{-1}$,

$\bar{b} = \bar{c}\, T^{1/2}[1-\phi_1]^{-1}$ and T is the sample size, $\underline{c}, \bar{c} \in R/\{0\}$ and $\underline{c} < \bar{c}$, $\phi_1 = \{\pm 0.1, \pm 0.2, \ldots, \pm 5\}$, $y_0 = y_{-1} = 0$, u_t are dependent error terms which achieved Beveridge-Nelson Decomposition Beveridge-Nelson (1981), ρ_1 and ρ_2 are the autoregressive coefficients.

2.1 Asymptotic Distributions of OLS Estimators under Different Tests of Hypothesis

Concepts of relative magnitude or order of magnitude are useful in investigating limiting behavior of random variables, where if $h(x)$ and $g(x)$ are two real functions that have a common domain $D \subset R$, and if the following relationship is exists for any positive constant $k\,(k > 0)$

$$\lim_{x \to x_0} \left| \frac{h(x)}{g(x)} \right| \le k \quad , \quad x \in (D - x_0)$$

Where, $h(x) = O(g(x))$. (2)

Schatzman (2002)

If $A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$ an $m \times n$ matrix with $r^* = rank(A)$ where B is $r^* \times r^*$ and invertible then the generalized inverse G for a given singular matrix A can be obtained as follows:

$$G = \begin{bmatrix} B^{-1} & 0 \\ 0 & 0 \end{bmatrix} \quad (3)$$

And if an equation represented as:

$$Ax = h \, , \; x \in R$$

Where, x is a vector or a matrix of unknown elements, h is vector or a matrix that has the same order as the product of Ax. So, to obtain the forms of unknown elements of x the following equation is need to be used:

$$x = Gh + (I - GA)z \; , \;\; z \in R \tag{4}$$

Where I is an identity matrix, z is a vector or a matrix of real numbers and G is the generalized inverse of the matrix A that satisfied $AGA = A$. Sawyer (2008)

If y_t is a pure random walk without drift as $y_t = y_{t-1} + u_t$, where $y_0 = y_{-1} = 0$, u_t are dependent error terms, and assume that u_t is defined as follows:

$$u_t = \phi_1 u_{t-1} + e_t = \psi(L)\, e_t = \sum_{j=0}^{\infty} \psi_j \, e_{t-j} \;\; , \left|\phi_1\right| < 1 \; , t = 1, 2, \ldots, T \tag{5}$$

Where:

$$\left.\begin{array}{l} E(e_t) = 0 \quad \text{for all } t \\ E(e_t \, e_s) = \begin{cases} \sigma^2 \;\; \text{if} \;\; t = s \\ 0 \quad \text{if} \quad t \neq s \end{cases} \\ \sum_{j=0}^{\infty} |j| \, \psi_j < \infty \end{array}\right\} \tag{6}$$

Then the following relationship exists:

$$\sum_{s=1}^{t} u_s = \psi(1)\sum_{s=1}^{t} e_s + \eta_t - \eta_0 \quad ,t=1,2,...,T \qquad (7)$$

$$\left.\begin{array}{l} \psi(1) = \sum_{j=0}^{\infty} \psi_j \\ \eta_t = \sum_{j=0}^{\infty} a_j\, e_{t-j} \\ a_j = -(\psi_{j+1} + \psi_{j+2} + \psi_{j+3} + \ldots) = -\sum_{i=1}^{\infty} \psi_{j+i} \;, \quad \sum_{j=0}^{\infty} \left| a_j \right| < \infty \\ \eta_0 = -(\psi_1 + \psi_2 + \psi_3 + \ldots) e_0 - (\psi_2 + \psi_3 + \psi_4 + \ldots) e_{-1} \\ \qquad - (\psi_3 + \psi_4 + \psi_5 + \ldots) e_{-2} + \ldots\ldots \end{array}\right\}$$

By defining the following quantities:

$$\left.\begin{array}{l} \gamma_j = E(u_t u_{t-j}) = \sigma^2 \sum_{s=0}^{\infty} \psi_s \psi_{s+j} \quad , j = 0,1,2,... \\ \lambda = \sigma \sum_{j=0}^{\infty} \psi_j = \sigma\, \psi(1) \\ y_t = u_1 + u_2 + \ldots + u_t \quad , t = 1, 2, \ldots, T \end{array}\right\} \qquad (8)$$

Then the following results are obtained:

$$\left.\begin{array}{l} 1)\; T^{-1} \sum_{t=1}^{T} u_t u_{t-j} \xrightarrow{p} \gamma_j \;, \; j = 0, 1, 2, \ldots \\ 2)\; T^{-1} \sum_{t=1}^{T} y_{t-1} u_t \xrightarrow{d} \tfrac{1}{2}\{\lambda^2 [W(1)_{\underline{c}}^{\bar{c}}]^2 - \gamma_0\} \\ 3)\; T^{-3/2} \sum_{t=1}^{T} y_{t-1} \xrightarrow{d} \lambda \int_0^1 W(r)_{\underline{c}}^{\bar{c}}\, dr \\ 4)\; T^{-2} \sum_{t=1}^{T} y_{t-1}^2 \xrightarrow{d} \lambda^2 \int_0^1 [W(r)_{\underline{c}}^{\bar{c}}]^2\, dr \end{array}\right\} \qquad (9)$$

Where, $W_{\underline{c}}^{\bar{c}}(r)$ is a Regulated Brownian Motion, when $r=1$, then:

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} u_t \xrightarrow{d} \sigma\psi(1) W_{\underline{c}}^{\bar{c}}(1) = \lambda W_{\underline{c}}^{\bar{c}}(1) \qquad (10)$$

By using equation (2) the results for orders of convergence of estimators in these equations will be as follows:

$$\left.\begin{array}{l} 1) T = O_p(T) \\ 2) \sum_{t=1}^{T} u_t = O_p(T^{1/2}) \\ 3) \sum_{t=1}^{T} u_t u_{t-j} = O_p(T) \\ 4) \sum_{t=1}^{T} y_{t-1} u_t = O_p(T) \\ 5) \sum_{t=1}^{T} y_{t-1} = O_p(T^{3/2}) \\ 6) \sum_{t=1}^{T} y_{t-1}^2 = O_p(T^2) \end{array}\right\}. \tag{11}$$

Amer (2015)

The asymptotic distributions of OLS estimators $\hat{\rho}_1$ and $\hat{\rho}_2$ for bounded AR (2) model without constant that represented by equation (1) for testing the null hypothesis $H_0: \rho_1 = 1, \rho_2 = 0$, (i.e. $y_t = y_{t-1} + u_t$) against the alternative hypothesis $H_a: |\rho_1| < 1, |\rho_2| < 1$, (i.e. $y_t = \rho_1 y_{t-1} + \rho_2 y_{t-2} + u_t$) will be derived as follows:

Lemma (1): If y_t is a pure random walk without drift as $y_t = y_{t-1} + u_t$, where $y_0 = y_{-1} = 0$, u_t are dependent error terms that achieved the Beveridge-Nelson Decomposition as in equation (7) then as $T \to \infty$ the following results are obtained:

$$\left.\begin{array}{l} 1) T^{-1} \sum_{t=1}^{T} y_{t-2} u_t \xrightarrow{d} \frac{1}{2}\{\lambda^2 [W_{\underline{c}}^{\bar{c}}(1)]^2 - \gamma_0\} - \gamma_1 \\ 2) T^{-3/2} \sum_{t=1}^{T} y_{t-2} \xrightarrow{d} \lambda \int_0^1 W_{\underline{c}}^{\bar{c}}(r) dr \\ 3) T^{-2} \sum_{t=1}^{T} y_{t-2}^2 \xrightarrow{d} \lambda^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2 dr \\ 4) T^{-2} \sum_{t=1}^{T} y_{t-1} y_{t-2} \xrightarrow{d} \lambda^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2 dr \end{array}\right\} \tag{12}$$

Where, $\gamma_0 = \sigma^2 \sum_{s=0}^{\infty} \psi_s^2 , \gamma_1 = \sigma^2 \sum_{s=0}^{\infty} \psi_s \psi_{s+1}$ and $\lambda = \sigma \sum_{j=0}^{\infty} \psi_j = \sigma \psi(1)$.

Proof:

Part (1)

From the successive substituting of y_t then:

$$y_{t-2} = y_{t-1} - u_{t-1} \tag{13}$$

So,

$$T^{-1} \sum_{t=1}^{T} y_{t-2} u_t = T^{-1} \sum_{t=1}^{T} y_{t-1} u_t - T^{-1} \sum_{t=1}^{T} u_{t-1} u_t \tag{14}$$

By using equation (9) then:

$$\left.\begin{aligned} &1) T^{-1} \sum_{t=1}^{T} y_{t-1} u_t \xrightarrow{d} \tfrac{1}{2} \{ \lambda^2 [W_{\underline{c}}^{\bar{c}}(1)]^2 - \gamma_0 \} \\ &2) T^{-1} \sum_{t=1}^{T} u_{t-1} u_t \xrightarrow{d} \gamma_1 \end{aligned}\right\} \tag{15}$$

Then, by substituting from equations (15) in (14) it can be concluded that:

$$T^{-1} \sum_{t=1}^{T} y_{t-2} u_t \xrightarrow{d} \tfrac{1}{2} \{ \lambda^2 [W_{\underline{c}}^{\bar{c}}(1)]^2 - \gamma_0 \} - \gamma_1$$

Part (2)

From equation (13);

$$T^{-3/2} \sum_{t=1}^{T} y_{t-2} = T^{-3/2} \sum_{t=1}^{T} y_{t-1} - T^{-3/2} \sum_{t=1}^{T} u_{t-1} \tag{16}$$

From equation (11) the order of convergence of $\sum_{t=1}^{T} u_{t-1} = O_P(T^{1/2})$ then:

$$T^{-3/2}\sum_{t=1}^{T}u_{t-1}\xrightarrow{d}0 \qquad (17)$$

By using equation (9) it can be concluded that:

$$T^{-3/2}\sum_{t=1}^{T}y_{t-1}\xrightarrow{d}\lambda\int_0^1 W_{\underline{c}}^{\bar{c}}(r)dr \qquad (18)$$

Then, by substituting from equations (17 & 18) in (16) it can be concluded that:

$$T^{-3/2}\sum_{t=1}^{T}y_{t-2}\xrightarrow{d}\lambda\int_0^1 W_{\underline{c}}^{\bar{c}}(r)dr \quad , \quad (y_{-1}=0)$$

Part (3)

From equation (13);

$$T^{-2}\sum_{t=1}^{T}y_{t-2}^2 = T^{-2}\sum_{t=1}^{T}y_{t-1}^2 - 2T^{-2}\sum_{t=1}^{T}y_{t-1}u_{t-1} + T^{-2}\sum_{t=1}^{T}u_{t-1}^2 \qquad (19)$$

From equation (11) the order of convergence of $\sum_{t=1}^{T}y_{t-1}^2 = O_P(T^2)$ and the order of convergence of $\sum_{t=1}^{T}u_{t-1}^2 = O_P(T)$ then:

$$\left.\begin{array}{l} 1)T^{-2}\sum_{t=1}^{T}y_{t-1}u_{t-1}\xrightarrow{d}0 \\ 2)T^{-2}\sum_{t=1}^{T}u_{t-1}^2\xrightarrow{d}0 \end{array}\right\} \qquad (20)$$

By using equation (9) then:

$$T^{-2}\sum_{t=1}^{T}y_{t-1}^2\xrightarrow{d}\lambda^2\int_0^1[W_{\underline{c}}^{\bar{c}}(r)]^2dr \qquad (21)$$

Then, by substituting from equations (20 & 21) in (19), it can be concluded that:

$$T^{-2}\sum_{t=1}^{T}y_{t-2}^2\xrightarrow{d}\lambda^2\int_0^1[W_{\underline{c}}^{\bar{c}}(r)]^2dr \quad , \quad (y_{-1}=0)$$

Part (4)

From equation (13);

$$T^{-2}\sum_{t=1}^{T} y_{t-1}\, y_{t-2} = T^{-2}\sum_{t=1}^{T} y_{t-1}^{2} - T^{-2}\sum_{t=1}^{T} y_{t-1} u_{t-1} \qquad (22)$$

Then, by substituting from equations (20 (1) & 21) in (22) it can be concluded that:

$$T^{-2}\sum_{t=1}^{T} y_{t-1}\, y_{t-2} \xrightarrow{d} \lambda^{2}\int_{0}^{1}[W_{\underline{c}}^{\bar{c}}(r)]^{2}\, dr$$

2.2 Asymptotic Distributions of OLS Estimators under Different Tests of Hypothesis

The asymptotic distributions of OLS estimators $\hat{\rho}_1$ and $\hat{\rho}_2$ for bounded AR (2) model that represented by equation (1) for testing the null hypothesis $H_0: \rho_1 = 1,\ \rho_2 = 0$, (i.e. $y_t = y_{t-1} + u_t$) against the alternative hypothesis $H_a: |\rho_1| < 1,\ |\rho_2| < 1$, (i.e. $y_t = \rho_1\, y_{t-1} + \rho_2\, y_{t-2} + u_t$) will be derived as follows:

Lemma (2): For model (1) and under the test $H_0: \rho_1 = 1$ and $\rho_2 = 0$, then the asymptotic distributions of $T(\hat{\rho}_1 - 1)$ and $T\hat{\rho}_2$ will be as follows:

$$1)\ T(\hat{\rho}_1 - 1) \xrightarrow{d} \frac{\frac{1}{2}\{\lambda^{2}[W_{\underline{c}}^{\bar{c}}(1)]^{2} - \gamma_0\}}{\lambda^{2}\int_{0}^{1}[W_{\underline{c}}^{\bar{c}}(r)]^{2}\, dr} - z_2$$

$$2)\ T\hat{\rho}_2 \xrightarrow{d} z_2\ , \qquad z_2 \in \underline{c}\sqrt{T} \text{ or } \bar{c}\sqrt{T}$$

Proof:

Model (1) can be rewritten in matrix form as follows:

$$Y^* = X^* \beta^* + \mathbf{u}^* \tag{23}$$

Where:

$$\beta^* = \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix},\ X^* = \begin{bmatrix} y_0 & y_{-1} \\ y_1 & y_0 \\ \vdots & \vdots \\ y_{T-1} & y_{T-2} \end{bmatrix},\ Y^* = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{bmatrix} \text{ and } \mathbf{u}^* = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_T \end{bmatrix}$$

Then the OLS Estimators of $\hat{\rho}_1$ and $\hat{\rho}_2$ for model (1) can be obtained as follows:

$$\hat{\beta}^* = (X^{*'} X^*)^{-1} X^{*'} Y^*$$

Under the null hypothesis that $H_0 : \rho_1 = 1 \text{ and } \rho_2 = 0$ or $\beta^{*'} = (1 \quad 0)$ and by using equation (23) then:

$$\hat{\beta}^* - \beta^* = (X^{*'} X^*)^{-1} X^{*'} \mathbf{u}^*$$

$$\begin{pmatrix} \hat{\rho}_1 - 1 \\ \hat{\rho}_2 \end{pmatrix} = \begin{pmatrix} \sum_{t=1}^T y_{t-1}^2 & \sum_{t=1}^T y_{t-1} y_{t-2} \\ \sum_{t=1}^T y_{t-1} y_{t-2} & \sum_{t=1}^T y_{t-2}^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum_{t=1}^T y_{t-1} u_t \\ \sum_{t=1}^T y_{t-2} u_t \end{pmatrix} \tag{24}$$

From equations (11 (4,6)), the order of convergence of $\sum_{t=1}^T y_{t-1} u_t$ and $\sum_{t=1}^T y_{t-1}^2$ $(\sum_{t=2}^T y_{t-2}^2)$ will be $O_p(T)$ and $O_p(T^2)$ respectively. Also, from equation (12 (1)) and by using the role of equation (2) then the order of convergence of $\sum_{t=1}^T y_{t-2} u_t = O_p(T)$, and from equation (12 (4)) and by using the role of equation (2) then the order of convergence of the order of convergence of $\sum_{t=1}^T y_{t-1} y_{t-2} = O_p(T^2)$.

Then, the order of convergence of elements in equation (24) will be as follows:

$$\begin{pmatrix} \hat{\rho}_1 - 1 \\ \hat{\rho}_2 \end{pmatrix} = \begin{pmatrix} O_p(T^2) & O_p(T^2) \\ O_p(T^2) & O_p(T^2) \end{pmatrix}^{-1} \begin{pmatrix} O_p(T) \\ O_p(T) \end{pmatrix}$$

Then, to obtain the asymptotic distributions of the estimators equation (24) will be multiplied by the following scaling matrix:

$$\psi''_T = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}$$

Then equation (24) will be as follows:

$$\psi''_T(\hat{\beta}^* - \beta^*) = \left\{ \psi''^{-1}_T (X^{*'} X^*) \psi''^{-1}_T \right\}^{-1} \psi''^{-1}_T X^{*'} \mathbf{u}^*$$

$$\begin{pmatrix} T(\hat{\rho}_1 - 1) \\ T\hat{\rho}_2 \end{pmatrix} = \begin{pmatrix} T^{-2}\sum_{t=1}^{T} y_{t-1}^2 & T^{-2}\sum_{t=1}^{T} y_{t-1}y_{t-2} \\ T^{-2}\sum_{t=1}^{T} y_{t-1}y_{t-2} & T^{-2}\sum_{t=1}^{T} y_{t-2}^2 \end{pmatrix}^{-1} \begin{pmatrix} T^{-1}\sum_{t=1}^{T} y_{t-1}u_t \\ T^{-1}\sum_{t=1}^{T} y_{t-2}u_t \end{pmatrix} \qquad (25)$$

From equations (9 (2,4)), $T^{-1}\sum_{t=1}^{T} y_{t-1}u_t$ and $T^{-2}\sum_{t=1}^{T} y_{t-1}^2$ convergence in distribution to $\frac{1}{2}\{\lambda^2[W(1)_{\underline{c}}^{\bar{c}}]^2 - \gamma_0\}$ and $\lambda^2\int_0^1 [W(r)_{\underline{c}}^{\bar{c}}]^2\, dr$ respectively. Also, from equation (12), $T^{-1}\sum_{t=1}^{T} y_{t-2}u_t$ and $T^{-2}\sum_{t=1}^{T} y_{t-2}^2$ $(T^{-2}\sum_{t=1}^{T} y_{t-1}y_{t-2})$ convergence in distribution to $\frac{1}{2}\{\lambda^2[W(1)_{\underline{c}}^{\bar{c}}]^2 - \gamma_0\} - \gamma_1$ and $\lambda^2\int_0^1 [W(r)_{\underline{c}}^{\bar{c}}]^2\, dr$ respectively.

Then, as $T \to \infty$ and by using the above results equation (25) will be as follows:

$x_7 = A_7 h_7$, $x_7 \in R_{(2\times1)}$ (i.e. vector of order (2×1) of real numbers) (26)

Where:

$$x_7 = \lim_{T\to\infty}\begin{pmatrix} T(\hat{\rho}_1 - 1) \\ T\hat{\rho}_2 \end{pmatrix},\ A_7 = \begin{pmatrix} \lambda^2 \int_0^1 [\bar{W}_{\underline{c}}^c(r)]^2\, dr & \lambda^2 \int_0^1 [\bar{W}_{\underline{c}}^c(r)]^2\, dr \\ \lambda^2 \int_0^1 [\bar{W}_{\underline{c}}^c(r)]^2\, dr & \lambda^2 \int_0^1 [\bar{W}_{\underline{c}}^c(r)]^2\, dr \end{pmatrix}$$

and $h_7 = \begin{pmatrix} \frac{1}{2}\{\lambda^2 [\bar{W}_{\underline{c}}^c(1)]^2 - \gamma_0\} \\ \frac{1}{2}\{\lambda^2 [\bar{W}_{\underline{c}}^c(1)]^2 - \gamma_0\} - \gamma_1 \end{pmatrix}$.

Since $|A_7| = 0$ a generalized inverse G_{71} of A_7 will obtained by using equation (3) and it will be as:

$$G_{71} = \begin{pmatrix} \dfrac{1}{\lambda^2 \int_0^1 [\bar{W}_{\underline{c}}^c(r)]^2\, dr} & 0 \\ 0 & 0 \end{pmatrix}_{(2\times 2)}$$

Now to obtain the forms of elements of x_7 in equation (26), equation (4) will be used as follows:

Since:

$$G_{71}A_7 = \begin{pmatrix} \dfrac{1}{\lambda^2 \int_0^1 [\bar{W}_{\underline{c}}^c(r)]^2\, dr} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \lambda^2 \int_0^1 [\bar{W}_{\underline{c}}^c(r)]^2\, dr & \lambda^2 \int_0^1 [\bar{W}_{\underline{c}}^c(r)]^2\, dr \\ \lambda^2 \int_0^1 [\bar{W}_{\underline{c}}^c(r)]^2\, dr & \lambda^2 \int_0^1 [\bar{W}_{\underline{c}}^c(r)]^2\, dr \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$G_{71}h_7 = \begin{pmatrix} \dfrac{1}{\lambda^2 \int_0^1 [\bar{W}_{\underline{c}}^c(r)]^2\, dr} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2}\{\lambda^2 [\bar{W}_{\underline{c}}^c(1)]^2 - \gamma_0\} \\ \frac{1}{2}\{\lambda^2 [\bar{W}_{\underline{c}}^c(1)]^2 - \gamma_0\} - \gamma_1 \end{pmatrix} = \begin{pmatrix} \dfrac{\frac{1}{2}\{\lambda^2 [\bar{W}_{\underline{c}}^c(1)]^2 - \gamma_0\}}{\lambda^2 \int_0^1 [\bar{W}_{\underline{c}}^c(r)]^2\, dr} \\ 0 \end{pmatrix}$$

Then, by using equation (4) it can be concluded that:

$$x_7 = \begin{pmatrix} \dfrac{\frac{1}{2}\{\lambda^2 [\bar{W}_{\underline{c}}^c(1)]^2 - \gamma_0\}}{\lambda^2 \int_0^1 [\bar{W}_{\underline{c}}^c(r)]^2\, dr} \\ 0 \end{pmatrix} + \left\{ I_2 - \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

Where $z's$ are real numbers, then the asymptotic distributions of $T(\hat{\rho}_1 - 1)$ and $T\hat{\rho}_2$ will be as follows:

$$\left.\begin{array}{l} 1)\, T(\hat{\rho}_1 - 1) \xrightarrow{d} \dfrac{\frac{1}{2}\{\lambda^2 [W_{\underline{c}}^{\bar{c}}(1)]^2 - \gamma_0\}}{\lambda^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\, dr} - z_2 \\ 2)\, T\hat{\rho}_2 \xrightarrow{d} z_2 \,, \qquad z_2 \in \underline{c}\sqrt{T} \text{ or } \bar{c}\sqrt{T} \end{array}\right\} \tag{27}$$

Corollary (1): If there is another generalized inverse G_{72} of A_7 that can be obtained by using equation (3), it will be as follows:

$$G_{72} = \begin{pmatrix} 0 & \dfrac{1}{\lambda^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\, dr} \\ 0 & 0 \end{pmatrix}_{(2\times 2)}$$

Then, the asymptotic distributions of $T(\hat{\rho}_1 - 1)$ and $T\hat{\rho}_2$ will be as follows:

$$\left.\begin{array}{l} 1)\, T(\hat{\rho}_1 - 1) \xrightarrow{d} \dfrac{\frac{1}{2}\{\lambda^2 [W_{\underline{c}}^{\bar{c}}(1)]^2 - \gamma_0\} - \gamma_1}{\lambda^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\, dr} - z_2 \\ 2)\, T\hat{\rho}_2 \xrightarrow{d} z_2 \,, \qquad z_2 \in \underline{c}\sqrt{T} \text{ or } \bar{c}\sqrt{T} \end{array}\right\} \tag{28}$$

Corollary (2): If there is another generalized inverse G_{73} of A_7 that can be obtained by using equation (3), it will be as follows:

$$G_{73} = \begin{pmatrix} 0 & 0 \\ \dfrac{1}{\lambda^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\, dr} & 0 \end{pmatrix}_{(2\times 2)}$$

Then, the asymptotic distributions of $T(\hat{\rho}_1 - 1)$ and $T\hat{\rho}_2$ will be as follows:

$$\left.\begin{array}{l} 1)\, T(\hat{\rho}_1 - 1) \xrightarrow{d} z_1 \\ 2)\, T\hat{\rho}_2 \xrightarrow{d} \dfrac{\frac{1}{2}\{\lambda^2 [W_{\underline{c}}^{\bar{c}}(1)]^2 - \gamma_0\}}{\lambda^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\, dr} - z_1 \;,\; z_1 \in \underline{c}\sqrt{T} \text{ or } \bar{c}\sqrt{T} \end{array}\right\} \qquad (29)$$

Corollary (3): If there is another generalized inverse G_{74} of A_7 that can be obtained by using equation (3), it will be as follows:

$$G_{74} = \begin{pmatrix} 0 & 0 \\ 0 & \dfrac{1}{\lambda^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\, dr} \end{pmatrix}_{(2\times 2)}$$

Then, the asymptotic distributions of $T(\hat{\rho}_1 - 1)$ and $T\hat{\rho}_2$ will be as follows:

$$\left.\begin{array}{l} 1)\, T(\hat{\rho}_1 - 1) \xrightarrow{d} z_1 \\ 2)\, T\hat{\rho}_2 \xrightarrow{d} \dfrac{\frac{1}{2}\{\lambda^2 [W_{\underline{c}}^{\bar{c}}(1)]^2 - \gamma_0\} - \gamma_1}{\lambda^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\, dr} - z_1 \;,\; z_1 \in \underline{c}\sqrt{T} \text{ or } \bar{c}\sqrt{T} \end{array}\right\} \qquad (30)$$

2.3 Asymptotic Distributions of the $t-type$ Statistics under Different Tests of Hypothesis

In addition to the previous tests in (2.2), the tests that based on $t-type$ statistics for the estimators $\hat{\rho}_1$ and $\hat{\rho}_2$ under the test $H_0 : \rho_1 = 1 \,,\, \rho_2 = 0$, $(y_t = y_{t-1} + u_t)$ against $H_a : |\rho_1| < 1 \,,\, |\rho_2| < 1$, (i.e. $y_t = \rho_1 y_{t-1} + \rho_2 y_{t-2} + u_t$) will be derived as follows:

Lemma (3): If the variance-covariance matrix of the estimators of model (1) under the null hypothesis $H_0 : \rho_1 = 1 \,,\, \rho_2 = 0$ that can be written in matrix form as follows:

$$Var(\hat{\beta}^*) = S_T'^2 (X^{*'} X^*)^{-1} \tag{31}$$

Such that,

$$\left.\begin{array}{l}
1) Var(\hat{\beta}^*) = \begin{pmatrix} Var(\hat{\rho}_1) & Cov(\hat{\rho}_1, \hat{\rho}_2) \\ Cov(\hat{\rho}_1, \hat{\rho}_2) & Var(\hat{\rho}_2) \end{pmatrix} \\
2) (X^{*'} X^*)^{-1} = \begin{pmatrix} \sum_{t=1}^{T} y_{t-1}^2 & \sum_{t=1}^{T} y_{t-1} y_{t-2} \\ \sum_{t=1}^{T} y_{t-1} y_{t-2} & \sum_{t=1}^{T} y_{t-2}^2 \end{pmatrix}^{-1} \\
3) S_T'^2 = \sum_{t=1}^{T} (y_t - \hat{\rho}_1 y_{t-1} - \hat{\rho}_2 y_{t-2})^2 / (T-2) = \sum_{t=1}^{T} \hat{u}_t^2 / (T-2)
\end{array}\right\} \tag{32}$$

Then, the asymptotic distributions for $t_{\hat{\rho}_1}$ and $t_{\hat{\rho}_2}$ will be follows:

$$1) t_{\hat{\rho}_1} = [T(\hat{\rho}_1 - 1)][T^2 Var(\hat{\rho}_1)]^{-1/2} \xrightarrow{d}$$

$$\left[\frac{\frac{1}{2}\{\lambda^2 [W_{\underline{c}}^{\bar{c}}(1)]^2 - \gamma_0\}}{\lambda^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2 \, dr} - z_2\right]\left[\frac{\gamma_0}{\lambda^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2 \, dr} - z_{21}\right]^{-\frac{1}{2}}$$

$$2) t_{\hat{\rho}_2} = [T\hat{\rho}_2][T^2 Var(\hat{\rho}_2)]^{-1/2} \xrightarrow{d} z_2 (z_{22})^{-1/2}, \; z_2 \in \underline{c}\sqrt{T} \text{ or } \bar{c}\sqrt{T}, z_{21} \in \underline{c}\sqrt{T}, \; z_{22} \in \bar{c}\sqrt{T}$$

Proof:

By multiplying equation (31) by the scaling matrix ψ_T'' as follows:

$$\psi_T'' Var(\hat{\beta}^*) \psi_T'' = S_T'^2 (\psi_T''^{-1} X^{*'} X^* \psi_T''^{-1})^{-1} \tag{33}$$

Where ψ_T'' is defined as in lemma (2).

And by substituting from equations (32 (1,2)) in equation (33) then the variance-covariance matrix will be as follows:

$$\begin{pmatrix} T^2 Var(\hat{\rho}_1) & T^2 Cov(\hat{\rho}_1, \hat{\rho}_2) \\ T^2 Cov(\hat{\rho}_1, \hat{\rho}_2) & T^2 Var(\hat{\rho}_2) \end{pmatrix} = S'^2_T B_4 \qquad (34)$$

Where:

$$B_4 = \begin{pmatrix} T^{-2} \sum_{t=1}^{T} y_{t-1}^2 & T^{-2} \sum_{t=1}^{T} y_{t-1} y_{t-2} \\ T^{-2} \sum_{t=1}^{T} y_{t-1} y_{t-2} & T^{-2} \sum_{t=1}^{T} y_{t-2}^2 \end{pmatrix}^{-1}$$

As $T \to \infty$ and from the weak law of large number, Bell (2015), and from equation (9 (1)) then the convergence in probability of S'^2_T will be as follows:

$$S'^2_T = \sum_{t=1}^{T} \hat{u}_t^2 / (T-2) \xrightarrow{p} \gamma_0 \qquad (35)$$

From equation (9 (4)), $T^{-2} \sum_{t=1}^{T} y_{t-1}^2 \xrightarrow{d} \lambda^2 \int_0^1 [W(r)_{\underline{c}}^{\bar{c}}]^2 \, dr$. Also, from equations (12 (3,4)), $T^{-2} \sum_{t=1}^{T} y_{t-2}^2 (T^{-2} \sum_{t=1}^{T} y_{t-1} y_{t-2})$ convergence in distribution to $\lambda^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2 \, dr$.

Then, as $T \to \infty$, by using the above results equation (34) will be as follows:

$$x_8 = A_8^{-1} h_8 \ , \ x_8 \in R_{(2\times 2)} \qquad (36)$$

Where:

$$x_8 = \lim_{T \to \infty} \begin{pmatrix} T^2 Var(\hat{\rho}_1) & T^2 Cov(\hat{\rho}_1, \hat{\rho}_2) \\ T^2 Cov(\hat{\rho}_1, \hat{\rho}_2) & T^2 Var(\hat{\rho}_2) \end{pmatrix}, A_8 = \begin{pmatrix} \frac{\lambda^2}{\gamma_0} \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2 \, dr & \frac{\lambda^2}{\gamma_0} \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2 \, dr \\ \frac{\lambda^2}{\gamma_0} \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2 \, dr & \frac{\lambda^2}{\gamma_0} \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2 \, dr \end{pmatrix}$$

, $h_8 = I_2$ and A_8 is the asymptotic distribution of the matrix $S_T'^2 B_4$.

Since $|A_8| = 0$ a generalized inverse G_{81} of A_8 will obtained by using equation (3) and it will be as:

$$G_{81} = \begin{pmatrix} \frac{\gamma_0}{\lambda^2 \int_0^1 [\bar{W}_{\underline{c}}^c(r)]^2 \, dr} & 0 \\ 0 & 0 \end{pmatrix}_{(2\times 2)}$$

Now to obtain the forms of elements of x_8 in equation (36), equation (4) will be used, the forms of the asymptotic distributions of $T^2 Var(\hat{\rho}_1)$ and $T^2 Var(\hat{\rho}_2)$ and the asymptotic distributions for $t_{\hat{\rho}_1}$ and $t_{\hat{\rho}_2}$ will be derived as follows:

Since:

$$G_{81}A_8 = \begin{pmatrix} \frac{\gamma_0}{\lambda^2 \int_0^1 [\bar{W}_{\underline{c}}^c(r)]^2 \, dr} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\lambda^2}{\gamma_0} \int_0^1 [\bar{W}_{\underline{c}}^c(r)]^2 \, dr & \frac{\lambda^2}{\gamma_0} \int_0^1 [\bar{W}_{\underline{c}}^c(r)]^2 \, dr \\ \frac{\lambda^2}{\gamma_0} \int_0^1 [\bar{W}_{\underline{c}}^c(r)]^2 \, dr & \frac{\lambda^2}{\gamma_0} \int_0^1 [\bar{W}_{\underline{c}}^c(r)]^2 \, dr \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$G_{81} h_8 = \begin{pmatrix} \frac{\gamma_0}{\lambda^2 \int_0^1 [\bar{W}_{\underline{c}}^c(r)]^2 \, dr} & 0 \\ 0 & 0 \end{pmatrix} I_2 = \begin{pmatrix} \frac{\gamma_0}{\lambda^2 \int_0^1 [\bar{W}_{\underline{c}}^c(r)]^2 \, dr} & 0 \\ 0 & 0 \end{pmatrix}$$

Then, by using equation (4) it can be concluded that:

$$x_8 = \begin{pmatrix} \frac{\gamma_0}{\lambda^2 \int_0^1 [\bar{W}_{\underline{c}}^c(r)]^2 \, dr} & 0 \\ 0 & 0 \end{pmatrix} + \left\{ I_2 - \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\} \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}$$

Where $z's$ are real numbers, then the asymptotic distributions of $T^2 Var(\hat{\rho}_1), T^2 Var(\hat{\rho}_2)$ will be as follows**:**

$$\left.\begin{aligned} &1) T^2 Var(\hat{\rho}_1) \xrightarrow{d} \frac{\gamma_0}{\lambda^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2 dr} - z_{21} > 0 \\ &2) T^2 Var(\hat{\rho}_2) \xrightarrow{d} z_{22} > 0 \ , \ z_{21} \in \underline{c}\sqrt{T} \ , \ z_{22} \in \bar{c}\sqrt{T} \end{aligned}\right\} \quad (37)$$

To achieve the variances in equation (37) to be positive, $z_{21} \leq 0$ and it assumed to be $z_{21} \in \underline{c}\sqrt{T}$ and $z_{22} > 0$ and it assumed to be $z_{22} \in \bar{c}\sqrt{T}$.

The $t-type$ statistics for the estimators $\hat{\rho}_1$ and $\hat{\rho}_2$ will be obtained as:

$$\left.\begin{aligned} &1) t_{\hat{\rho}_1} = [T(\hat{\rho}_1 - 1)][T^2 Var(\hat{\rho}_1)]^{-1/2} \\ &2) t_{\hat{\rho}_2} = [T\hat{\rho}_2][T^2 Var(\hat{\rho}_2)]^{-1/2} \end{aligned}\right\} \quad (38)$$

Then, by substituting from equation (27) that contains the asymptotic distributions of OLS estimators $T(\hat{\rho}_1 - 1)$ and $T\hat{\rho}_2$, and (37) in equation (38) then the asymptotic distributions for $t_{\hat{\rho}_1}$ and $t_{\hat{\rho}_2}$ respectively will be:

$$\left.\begin{array}{l} 1)\, t_{\hat{\rho}_1} = [T(\hat{\rho}_1 - 1)][T^2 Var(\hat{\rho}_1)]^{-1/2} \xrightarrow{d} \\ \quad \left[\frac{\frac{1}{2}\{\lambda^2 [W_{\underline{c}}^{\bar{c}}(1)]^2 - \gamma_0\}}{\lambda^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\, dr} - z_2\right]\left[\frac{\gamma_0}{\lambda^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\, dr} - z_{21}\right]^{-\frac{1}{2}} \\ 2)\, t_{\hat{\rho}_2} = [T\hat{\rho}_2][T^2 Var(\hat{\rho}_2)]^{-1/2} \xrightarrow{d} z_2 (z_{22})^{-1/2}, \\ \quad z_2 \in \underline{c}\sqrt{T} \text{ or } \bar{c}\sqrt{T},\, z_{21} \in \underline{c}\sqrt{T},\, z_{22} \in \bar{c}\sqrt{T} \end{array}\right\} \quad (39)$$

Corollary (4): If there is another generalized inverse G_{82} of A_8 that can be obtained by using equation (3), it will be as follows:

$$G_{82} = \begin{pmatrix} 0 & \dfrac{\gamma_0}{\lambda^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\, dr} \\ 0 & 0 \end{pmatrix}_{(2\times 2)}$$

Then, the asymptotic distributions for $t_{\hat{\rho}_1}$ and $t_{\hat{\rho}_2}$ will be:

$$\left.\begin{array}{l} 1)\, t_{\hat{\rho}_1} = [T(\hat{\rho}_1 - 1)][T^2 Var(\hat{\rho}_1)]^{-1/2} \xrightarrow{d} \left[\frac{\frac{1}{2}\{\lambda^2 [W_{\underline{c}}^{\bar{c}}(1)]^2 - \gamma_0\} - \gamma_1}{\lambda^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\, dr} - z_2\right](-z_{21})^{-1/2} \\ 2)\, t_{\hat{\rho}_2} = [T\hat{\rho}_2][T^2 Var(\hat{\rho}_2)]^{-1/2} \xrightarrow{d} z_2 (z_{22})^{-1/2}, \\ \quad z_2 \in \underline{c}\sqrt{T} \text{ or } \bar{c}\sqrt{T},\, z_{21} \in \underline{c}\sqrt{T},\, z_{22} \in \bar{c}\sqrt{T} \end{array}\right\} \quad (40)$$

Corollary (5): If there is another generalized inverse G_{83} of A_8 **that** can be obtained by using equation (3), it will be as follows:

$$G_{83} = \begin{pmatrix} 0 & 0 \\ \dfrac{\gamma_0}{\lambda^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\, dr} & 0 \end{pmatrix}_{(2\times 2)}$$

Then, the asymptotic distributions for $t_{\hat{\rho}_1}$ and $t_{\hat{\rho}_2}$ will be:

$$\left.\begin{array}{l} 1)\, t_{\hat{\rho}_1} = [T(\hat{\rho}_1 - 1)][T^2 Var(\hat{\rho}_1)]^{-1/2} \xrightarrow{d} z_1 (z_{11})^{-1/2} \\ 2)\, t_{\hat{\rho}_2} = [T\hat{\rho}_2][T^2 Var(\hat{\rho}_2)]^{-1/2} \xrightarrow{d} \left[\dfrac{\frac{1}{2}\{\lambda^2 [W_{\underline{c}}^{\bar{c}}(1)]^2 - \gamma_0\}}{\lambda^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2 dr} - z_1 \right] (-z_{12})^{-1/2} \\ \quad , z_1 \in \underline{c}\sqrt{T} \text{ or } \bar{c}\sqrt{T}, z_{11} \in \bar{c}\sqrt{T}, z_{12} \in \underline{c}\sqrt{T} \end{array}\right\} \quad (41)$$

Corollary (6): If there is another generalized inverse G_{84} of A_8 that can be obtained by using equation (3), it will be as follows:

$$G_{84} = \begin{pmatrix} 0 & 0 \\ 0 & \dfrac{\gamma_0}{\lambda^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2 dr} \end{pmatrix}_{(2\times 2)}$$

Then, the asymptotic distributions for $t_{\hat{\rho}_1}$ and $t_{\hat{\rho}_2}$ will be:

$$\left.\begin{array}{l} 1)\, t_{\hat{\rho}_1} = [T(\hat{\rho}_1 - 1)][T^2 Var(\hat{\rho}_1)]^{-1/2} \xrightarrow{d} z_1 (z_{11})^{-1/2} \\ 2)\, t_{\hat{\rho}_2} = [T\hat{\rho}_2][T^2 Var(\hat{\rho}_2)]^{-1/2} \xrightarrow{d} \\ \quad \left[\dfrac{\frac{1}{2}\{\lambda^2 [W_{\underline{c}}^{\bar{c}}(1)]^2 - \gamma_0\} - \gamma_1}{\lambda^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2 dr} - z_1 \right] \left[\dfrac{\gamma_0}{\lambda^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2 dr} - z_{12} \right]^{-1/2} \\ \quad , z_1 \in \underline{c}\sqrt{T} \text{ or } \bar{c}\sqrt{T}, z_{11} \in \bar{c}\sqrt{T}, z_{12} \in \underline{c}\sqrt{T} \end{array}\right\} \quad (42)$$

3. Simulation Study

A simulation study is used to obtain bias, MSE, Thiel's U under the null hypothesis $H_0: y_t = y_{t-1} + u_t$. Also, the same measures and the power of the test under the alternative hypothesis $H_a: y_t = \rho_1 y_{t-1} + \rho_2 y_{t-2} + u_t$ will be obtained in case of five samples size T = 30, 50, 100, 200 and 500 for five boundaries value $\bar{c} = -\underline{c} = 0.3, 0.5, 0.7, 0.9 \text{ and } 1.1$ and in case of ten values for the coefficient of dependent errors $\phi_1 = \pm 0.5, \pm 0.4, \pm 0.3, \pm 0.2 \text{ and } \pm 0.1$ by 5000 replications, OLS estimators of bounded AR (2) model without constant and with dependent errors which obtained in lemma (2) that used the generalized inverse G_{71}, in corollary (1) that used the generalized inverse G_{72}, in corollary (2) that used the generalized inverse G_{73} and in corollary (3) that used the generalized inverse G_{74} are used to obtain the bias, MSE, Thiel's U and the power of the test and the results can be discussed for the next five cases:

Case (1): T = 30

Table (1)

<table>
<tr><th colspan="2">$\bar{c}=-\underline{c}$</th><th>ϕ_1 / Criteria</th><th>0.5</th><th>0.4</th><th>0.3</th><th>0.2</th><th>0.1</th><th>-0.1</th><th>-0.2</th><th>-0.3</th><th>-0.4</th><th>-0.5</th></tr>
<tr><td rowspan="10">G_{71}</td><td rowspan="2">0.3</td><td>MSE</td><td colspan="7">H_a</td><td colspan="3">H_0</td></tr>
<tr><td>Thiel's U</td><td colspan="10" rowspan="9">H_a</td></tr>
<tr><td rowspan="2">0.5</td><td>MSE</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="2">0.7</td><td>MSE</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="2">0.9</td><td>MSE</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="2">1.1</td><td>MSE</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="10">G_{72}</td><td rowspan="2">0.3</td><td>MSE</td><td colspan="6" rowspan="2">H_a</td><td colspan="4" rowspan="2">H_0</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="2">0.5</td><td>MSE</td><td colspan="9">H_a</td><td>H_0</td></tr>
<tr><td>Thiel's U</td><td colspan="10" rowspan="7">H_a</td></tr>
<tr><td rowspan="2">0.7</td><td>MSE</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="2">0.9</td><td>MSE</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="2">1.1</td><td>MSE</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="10">G_{73}</td><td rowspan="2">0.3</td><td>MSE</td><td colspan="6">H_a</td><td colspan="4">H_0</td></tr>
<tr><td>Thiel's U</td><td colspan="3">H_a</td><td colspan="7">H_0</td></tr>
<tr><td rowspan="2">0.5</td><td>MSE</td><td colspan="7">H_a</td><td colspan="3">H_0</td></tr>
<tr><td>Thiel's U</td><td colspan="5">H_a</td><td colspan="5">H_0</td></tr>
<tr><td rowspan="2">0.7</td><td>MSE</td><td colspan="9">H_a</td><td>H_0</td></tr>
<tr><td>Thiel's U</td><td colspan="6">H_a</td><td colspan="4">H_0</td></tr>
<tr><td rowspan="2">0.9</td><td>MSE</td><td colspan="10">H_a</td></tr>
<tr><td>Thiel's U</td><td colspan="7">H_a</td><td colspan="3">H_0</td></tr>
<tr><td rowspan="2">1.1</td><td>MSE</td><td colspan="10">H_a</td></tr>
<tr><td>Thiel's U</td><td colspan="9">H_a</td><td>H_0</td></tr>
<tr><td rowspan="10">G_{74}</td><td rowspan="2">0.3</td><td>MSE</td><td colspan="8">H_a</td><td colspan="2">H_0</td></tr>
<tr><td>Thiel's U</td><td colspan="10">H_0</td></tr>
<tr><td rowspan="2">0.5</td><td>MSE</td><td colspan="9">H_a</td><td>H_0</td></tr>
<tr><td>Thiel's U</td><td colspan="5">H_a</td><td colspan="5">H_0</td></tr>
<tr><td rowspan="2">0.7</td><td>MSE</td><td colspan="10">H_a</td></tr>
<tr><td>Thiel's U</td><td colspan="6">H_a</td><td colspan="4">H_0</td></tr>
<tr><td rowspan="2">0.9</td><td>MSE</td><td colspan="10">H_a</td></tr>
<tr><td>Thiel's U</td><td colspan="8">H_a</td><td colspan="2">H_0</td></tr>
<tr><td rowspan="2">1.1</td><td>MSE</td><td colspan="10" rowspan="2">H_a</td></tr>
<tr><td>Thiel's U</td></tr>
</table>

It can be notice from table (1) that G_{71} approve the alternative hypothesis H_a for all values of $\bar{c}=-\underline{c}$ except for values of MSE and $\bar{c}=-\underline{c}=0.3$ in case of $\phi_1=-0.3,-0.4$ and -0.5, G_{72} approve the alternative hypothesis H_a for all values of $\bar{c}=-\underline{c}$ except for the values of MSE, Thiel's U and $\bar{c}=-\underline{c}=0.3$ in case of $\phi_1=-0.2,-0.3,-0.4$ and -0.5 and for the values of MSE and $\bar{c}=-\underline{c}=0.5$ in case of $\phi_1=-0.5$, G_{73} approve the alternative hypothesis H_a for most values of $\bar{c}=-\underline{c}$ except for the values of Thiel's U and $\bar{c}=-\underline{c}=0.3$ in case of negative values of ϕ_1 and G_{74} approve the alternative hypothesis H_a for most values of $\bar{c}=-\underline{c}$ except for the values of Thiel's U and $\bar{c}=-\underline{c}=0.3$ in case of all values of ϕ_1 and for the values of Thiel's U and $\bar{c}=-\underline{c}=0.5$ in case of negative values of ϕ_1. Also, the values of the power of the test are equal to integer one for both G_{71}, G_{72}, G_{73} and G_{74} for most values of $\bar{c}=-\underline{c}$ and ϕ_1 and the power of the test approve the alternative hypothesis H_a for all values of $\bar{c}=-\underline{c}$ and ϕ_1.

Case (2): T = 50

Table (2)

<table>
<tr><th colspan="2">$\bar{c} = -\underline{c}$</th><th>ϕ_1 / Criteria</th><th>0.5</th><th>0.4</th><th>0.3</th><th>0.2</th><th>0.1</th><th>-0.1</th><th>-0.2</th><th>-0.3</th><th>-0.4</th><th>-0.5</th></tr>
<tr><td rowspan="10">G_{71}</td><td rowspan="2">0.3</td><td>MSE</td><td colspan="8" rowspan="2">H_a</td><td colspan="2" rowspan="2">H_0</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="2">0.5</td><td>MSE</td><td colspan="10" rowspan="8">H_a</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="2">0.7</td><td>MSE</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="2">0.9</td><td>MSE</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="2">1.1</td><td>MSE</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="10">G_{72}</td><td rowspan="2">0.3</td><td>MSE</td><td colspan="6" rowspan="2">H_a</td><td colspan="4" rowspan="2">H_0</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="2">0.5</td><td>MSE</td><td colspan="9">H_a</td><td>H_0</td></tr>
<tr><td>Thiel's U</td><td colspan="10" rowspan="7">H_a</td></tr>
<tr><td rowspan="2">0.7</td><td>MSE</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="2">0.9</td><td>MSE</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="2">1.1</td><td>MSE</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="10">G_{73}</td><td rowspan="2">0.3</td><td>MSE</td><td colspan="7">H_a</td><td colspan="3">H_0</td></tr>
<tr><td>Thiel's U</td><td colspan="2">H_a</td><td colspan="8">H_0</td></tr>
<tr><td rowspan="2">0.5</td><td>MSE</td><td colspan="8">H_a</td><td colspan="2">H_0</td></tr>
<tr><td>Thiel's U</td><td colspan="5">H_a</td><td colspan="5">H_0</td></tr>
<tr><td rowspan="2">0.7</td><td>MSE</td><td colspan="10">H_a</td></tr>
<tr><td>Thiel's U</td><td colspan="6">H_a</td><td colspan="4">H_0</td></tr>
<tr><td rowspan="2">0.9</td><td>MSE</td><td colspan="10">H_a</td></tr>
<tr><td>Thiel's U</td><td colspan="8">H_a</td><td colspan="2">H_0</td></tr>
<tr><td rowspan="2">1.1</td><td>MSE</td><td colspan="10" rowspan="2">H_a</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="10">G_{74}</td><td rowspan="2">0.3</td><td>MSE</td><td colspan="9">H_a</td><td>H_0</td></tr>
<tr><td>Thiel's U</td><td colspan="10">H_0</td></tr>
<tr><td rowspan="2">0.5</td><td>MSE</td><td colspan="10">H_a</td></tr>
<tr><td>Thiel's U</td><td colspan="5">H_a</td><td colspan="5">H_0</td></tr>
<tr><td rowspan="2">0.7</td><td>MSE</td><td colspan="10">H_a</td></tr>
<tr><td>Thiel's U</td><td colspan="7">H_a</td><td colspan="3">H_0</td></tr>
<tr><td rowspan="2">0.9</td><td>MSE</td><td colspan="10">H_a</td></tr>
<tr><td>Thiel's U</td><td colspan="9">H_a</td><td>H_0</td></tr>
<tr><td rowspan="2">1.1</td><td>MSE</td><td colspan="10" rowspan="2">H_a</td></tr>
<tr><td>Thiel's U</td></tr>
</table>

It can be notice from table (2) that G_{71} approve the alternative hypothesis H_a for all values of $\bar{c}=-\underline{c}$ except for values of MSE, Thiel's U and $\bar{c}=-\underline{c}=0.3$ in case of $\phi_1=-0.4$ and -0.5, G_{72} approve the alternative hypothesis H_a for all values of $\bar{c}=-\underline{c}$ except for the values of MSE, Thiel's U and $\bar{c}=-\underline{c}=0.3$ in case of $\phi_1=-0.2, -0.3, -0.4$ and -0.5 and for the values of MSE and $\bar{c}=-\underline{c}=0.5$ in case of $\phi_1=-0.5$, G_{73} approve the alternative hypothesis H_a for most values of $\bar{c}=-\underline{c}$ except for the values of Thiel's U and $\bar{c}=-\underline{c}=0.3$ and 0.5 in case of negative values of ϕ_1 and G_{74} approve the alternative hypothesis H_a for most values of $\bar{c}=-\underline{c}$ except for the values of Thiel's U and $\bar{c}=-\underline{c}=0.3$ in case of all values of ϕ_1 and for the values of Thiel's U and $\bar{c}=-\underline{c}=0.5$ in case of negative values of ϕ_1. Also, the values of the power of the test are equal to integer one for both G_{71}, G_{72}, G_{73} and G_{74} and approve the alternative hypothesis H_a for all values of $\bar{c}=-\underline{c}$ and ϕ_1.

Case (3): T = 100

Table (3)

<table>
<tr><th colspan="2">$\bar{c}=-\underline{c}$</th><th>ϕ_1 / Criteria</th><th>0.5</th><th>0.4</th><th>0.3</th><th>0.2</th><th>0.1</th><th>-0.1</th><th>-0.2</th><th>-0.3</th><th>-0.4</th><th>-0.5</th></tr>
<tr><td rowspan="10">G_{71}</td><td rowspan="2">0.3</td><td>MSE</td><td colspan="7" rowspan="2">H_a</td><td colspan="3" rowspan="2">H_0</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="2">0.5</td><td>MSE</td><td colspan="10" rowspan="8">H_a</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="2">0.7</td><td>MSE</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="2">0.9</td><td>MSE</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="2">1.1</td><td>MSE</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="10">G_{72}</td><td rowspan="2">0.3</td><td>MSE</td><td colspan="6" rowspan="2">H_a</td><td colspan="4" rowspan="2">H_0</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="2">0.5</td><td>MSE</td><td colspan="9" rowspan="2">H_a</td><td>H_0</td></tr>
<tr><td>Thiel's U</td><td>H_0</td></tr>
<tr><td rowspan="2">0.7</td><td>MSE</td><td colspan="10" rowspan="6">H_a</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="2">0.9</td><td>MSE</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="2">1.1</td><td>MSE</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="10">G_{73}</td><td rowspan="2">0.3</td><td>MSE</td><td colspan="7">H_a</td><td colspan="3">H_0</td></tr>
<tr><td>Thiel's U</td><td colspan="10">H_0</td></tr>
<tr><td rowspan="2">0.5</td><td>MSE</td><td colspan="9">H_a</td><td>H_0</td></tr>
<tr><td>Thiel's U</td><td colspan="5">H_a</td><td colspan="5">H_0</td></tr>
<tr><td rowspan="2">0.7</td><td>MSE</td><td colspan="10">H_a</td></tr>
<tr><td>Thiel's U</td><td colspan="7">H_a</td><td colspan="3">H_0</td></tr>
<tr><td rowspan="2">0.9</td><td>MSE</td><td colspan="10">H_a</td></tr>
<tr><td>Thiel's U</td><td colspan="9">H_a</td><td>H_0</td></tr>
<tr><td rowspan="2">1.1</td><td>MSE</td><td colspan="10" rowspan="2">H_a</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="10">G_{74}</td><td rowspan="2">0.3</td><td>MSE</td><td colspan="8">H_a</td><td colspan="2">H_0</td></tr>
<tr><td>Thiel's U</td><td colspan="10">H_0</td></tr>
<tr><td rowspan="2">0.5</td><td>MSE</td><td colspan="10">H_a</td></tr>
<tr><td>Thiel's U</td><td colspan="5">H_a</td><td colspan="5">H_0</td></tr>
<tr><td rowspan="2">0.7</td><td>MSE</td><td colspan="10">H_a</td></tr>
<tr><td>Thiel's U</td><td colspan="7">H_a</td><td colspan="3">H_0</td></tr>
<tr><td rowspan="2">0.9</td><td>MSE</td><td colspan="10" rowspan="4">H_a</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="2">1.1</td><td>MSE</td></tr>
<tr><td>Thiel's U</td></tr>
</table>

It can be notice from table (3) that G_{71} approve the alternative hypothesis H_a for all values of $\bar{c}=-\underline{c}$ except for values of MSE, Thiel's U and $\bar{c}=-\underline{c}=0.3$ in case of $\phi_1=-0.3, -0.4$ and -0.5, G_{72} approve the alternative hypothesis H_a for all values of $\bar{c}=-\underline{c}$ except for the values of MSE, Thiel's U and $\bar{c}=-\underline{c}=0.3$ in case of $\phi_1=-0.2, -0.3, -0.4$ and -0.5 and for the values of MSE, Thiel's U and $\bar{c}=-\underline{c}=0.5$ in case of $\phi_1=-0.5$ and G_{73} and G_{74} approve the alternative hypothesis H_a for most values of $\bar{c}=-\underline{c}$ except for the values of Thiel's U and $\bar{c}=-\underline{c}=0.3$ in case of all values of ϕ_1 and for the values of Thiel's U and $\bar{c}=-\underline{c}=0.5$ in case of negative values of ϕ_1. Also, the values of the power of the test are equal to integer one for both G_{71}, G_{72}, G_{73} and G_{74} and approve the alternative hypothesis H_a for all values of $\bar{c}=-\underline{c}$ and ϕ_1.

Case (4): T = 200

Table (4)

<table>
<tr><th colspan="2">$\bar{c}=-\underline{c}$</th><th>ϕ_1 / Criteria</th><th>0.5</th><th>0.4</th><th>0.3</th><th>0.2</th><th>0.1</th><th>-0.1</th><th>-0.2</th><th>-0.3</th><th>-0.4</th><th>-0.5</th></tr>
<tr><td rowspan="10">G_{71}</td><td rowspan="2">0.3</td><td>MSE</td><td colspan="8">H_a</td><td colspan="2">H_0</td></tr>
<tr><td>Thiel's U</td><td colspan="6">H_a</td><td colspan="4">H_0</td></tr>
<tr><td rowspan="2">0.5</td><td>MSE</td><td colspan="10" rowspan="8">H_a</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="2">0.7</td><td>MSE</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="2">0.9</td><td>MSE</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="2">1.1</td><td>MSE</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="10">G_{72}</td><td rowspan="2">0.3</td><td>MSE</td><td colspan="7" rowspan="2">H_a</td><td colspan="3" rowspan="2">H_0</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="2">0.5</td><td>MSE</td><td colspan="10">H_a</td></tr>
<tr><td>Thiel's U</td><td colspan="9">H_a</td><td>H_0</td></tr>
<tr><td rowspan="2">0.7</td><td>MSE</td><td colspan="10" rowspan="6">H_a</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="2">0.9</td><td>MSE</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="2">1.1</td><td>MSE</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="10">G_{73}</td><td rowspan="2">0.3</td><td>MSE</td><td colspan="6">H_a</td><td colspan="4">H_0</td></tr>
<tr><td>Thiel's U</td><td colspan="10">H_0</td></tr>
<tr><td rowspan="2">0.5</td><td>MSE</td><td colspan="9">H_a</td><td>H_0</td></tr>
<tr><td>Thiel's U</td><td colspan="5">H_a</td><td colspan="5">H_0</td></tr>
<tr><td rowspan="2">0.7</td><td>MSE</td><td colspan="10">H_a</td></tr>
<tr><td>Thiel's U</td><td colspan="7">H_a</td><td colspan="3">H_0</td></tr>
<tr><td rowspan="2">0.9</td><td>MSE</td><td colspan="10">H_a</td></tr>
<tr><td>Thiel's U</td><td colspan="9">H_a</td><td>H_0</td></tr>
<tr><td rowspan="2">1.1</td><td>MSE</td><td colspan="10" rowspan="2">H_a</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="10">G_{74}</td><td rowspan="2">0.3</td><td>MSE</td><td colspan="10" rowspan="2">H_0</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="2">0.5</td><td>MSE</td><td colspan="10">H_a</td></tr>
<tr><td>Thiel's U</td><td colspan="5">H_a</td><td colspan="5">H_0</td></tr>
<tr><td rowspan="2">0.7</td><td>MSE</td><td colspan="10">H_a</td></tr>
<tr><td>Thiel's U</td><td colspan="8">H_a</td><td colspan="2">H_0</td></tr>
<tr><td rowspan="2">0.9</td><td>MSE</td><td colspan="10" rowspan="4">H_a</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="2">1.1</td><td>MSE</td></tr>
<tr><td>Thiel's U</td></tr>
</table>

It can be notice from table (4) that G_{71} approve the alternative hypothesis H_a for all values of $\bar{c}=-\underline{c}$ except for values of MSE and $\bar{c}=-\underline{c}=0.3$ in case of $\phi_1=-0.4$ and -0.5 and for values of Thiel's U and $\bar{c}=-\underline{c}=0.3$ in case of $\phi_1=-0.2,-0.3,-0.4$ and -0.5, G_{72} approve the alternative hypothesis H_a for all values of $\bar{c}=-\underline{c}$ except for the values of MSE, Thiel's U and $\bar{c}=-\underline{c}=0.3$ in case of $\phi_1=-0.3,-0.4$ and -0.5 and for the values of Thiel's U and $\bar{c}=-\underline{c}=0.5$ in case of $\phi_1=-0.5$, G_{73} approve the alternative hypothesis H_a for most values of $\bar{c}=-\underline{c}$ except for the values of Thiel's U and $\bar{c}=-\underline{c}=0.3$ in case of all values of ϕ_1 and for the values of Thiel's U and $\bar{c}=-\underline{c}=0.5$ in case of negative values of ϕ_1 and G_{74} approve the alternative hypothesis H_a for most values of $\bar{c}=-\underline{c}$ except for the values of MSE, Thiel's U and $\bar{c}=-\underline{c}=0.3$ in case of all values of ϕ_1 and for the values of Thiel's U and $\bar{c}=-\underline{c}=0.5$ in case of negative values of ϕ_1. Also, the values of the power of the test are equal to integer one for both G_{71}, G_{72}, G_{73} and G_{74} and approve the alternative hypothesis H_a for all values of $\bar{c}=-\underline{c}$ and ϕ_1.

Case (5): T = 500

Table (5)

<table>
<tr><th colspan="2">$\bar{c}=-\underline{c}$</th><th>ϕ_1 / Criteria</th><th>0.5</th><th>0.4</th><th>0.3</th><th>0.2</th><th>0.1</th><th>-0.1</th><th>-0.2</th><th>-0.3</th><th>-0.4</th><th>-0.5</th></tr>
<tr><td rowspan="10">G_{71}</td><td rowspan="2">0.3</td><td>MSE</td><td colspan="8">H_a</td><td colspan="2">H_0</td></tr>
<tr><td>Thiel's U</td><td colspan="6">H_a</td><td colspan="4">H_0</td></tr>
<tr><td rowspan="2">0.5</td><td>MSE</td><td colspan="10">H_a</td></tr>
<tr><td>Thiel's U</td><td colspan="9">H_a</td><td>H_0</td></tr>
<tr><td rowspan="2">0.7</td><td>MSE</td><td colspan="10" rowspan="6">H_a</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="2">0.9</td><td>MSE</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="2">1.1</td><td>MSE</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="10">G_{72}</td><td rowspan="2">0.3</td><td>MSE</td><td colspan="8">H_a</td><td colspan="2">H_0</td></tr>
<tr><td>Thiel's U</td><td colspan="6">H_a</td><td colspan="4">H_0</td></tr>
<tr><td rowspan="2">0.5</td><td>MSE</td><td colspan="10">H_a</td></tr>
<tr><td>Thiel's U</td><td colspan="8">H_a</td><td colspan="2">H_0</td></tr>
<tr><td rowspan="2">0.7</td><td>MSE</td><td colspan="10" rowspan="6">H_a</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="2">0.9</td><td>MSE</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="2">1.1</td><td>MSE</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="10">G_{73}</td><td rowspan="2">0.3</td><td>MSE</td><td colspan="10" rowspan="2">H_a</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="2">0.5</td><td>MSE</td><td colspan="8">H_a</td><td colspan="2">H_0</td></tr>
<tr><td>Thiel's U</td><td colspan="4">H_a</td><td colspan="6">H_0</td></tr>
<tr><td rowspan="2">0.7</td><td>MSE</td><td colspan="10">H_a</td></tr>
<tr><td>Thiel's U</td><td colspan="8">H_a</td><td colspan="2">H_0</td></tr>
<tr><td rowspan="2">0.9</td><td>MSE</td><td colspan="10" rowspan="4">H_a</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="2">1.1</td><td>MSE</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="10">G_{74}</td><td rowspan="2">0.3</td><td>MSE</td><td colspan="10" rowspan="2">H_a</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="2">0.5</td><td>MSE</td><td colspan="9">H_a</td><td>H_0</td></tr>
<tr><td>Thiel's U</td><td colspan="4">H_a</td><td colspan="6">H_0</td></tr>
<tr><td rowspan="2">0.7</td><td>MSE</td><td colspan="10">H_a</td></tr>
<tr><td>Thiel's U</td><td colspan="9">H_a</td><td>H_0</td></tr>
<tr><td rowspan="2">0.9</td><td>MSE</td><td colspan="10" rowspan="4">H_a</td></tr>
<tr><td>Thiel's U</td></tr>
<tr><td rowspan="2">1.1</td><td>MSE</td></tr>
<tr><td>Thiel's U</td></tr>
</table>

It can be notice from table (5) that G_{71} approve the alternative hypothesis H_a for all values of $\bar{c}=-\underline{c}$ except for values of MSE and $\bar{c}=-\underline{c}=0.3$ in case of $\phi_1=-0.4$ and -0.5, except for values of Thiel's U and $\bar{c}=-\underline{c}=0.3$ in case of $\phi_1=-0.2,-0.3,-0.4$ and -0.5 and except for values of Thiel's U and $\bar{c}=-\underline{c}=0.5$ in case of $\phi_1=-0.5$, G_{72} approve the alternative hypothesis H_a for all values of $\bar{c}=-\underline{c}$ except for values of MSE and $\bar{c}=-\underline{c}=0.3$ in case of $\phi_1=-0.4$ and -0.5, for values of Thiel's U and $\bar{c}=-\underline{c}=0.3$ in case of $\phi_1=-0.2,-0.3,-0.4$ and -0.5 and for values of Thiel's U and $\bar{c}=-\underline{c}=0.5$ in case of $\phi_1=-0.4$ and -0.5 and G_{73} and G_{74} approve the alternative hypothesis H_a for most values of $\bar{c}=-\underline{c}$ except for the values of Thiel's U and $\bar{c}=-\underline{c}=0.5$ in case of negative values of ϕ_1. Also, the values of the power of the test are equal to integer one for both G_{71}, G_{72}, G_{73} and G_{74} and approve the alternative hypothesis H_a for all values of $\bar{c}=-\underline{c}$ and ϕ_1.

4. Conclusions

Part (1): Theory

- The asymptotic distributions of OLS estimators of bounded AR (2) model without constant and with dependent errors under different tests of hypothesis have been derived.

- The asymptotic distributions of the $t-type$ statistics of OLS estimators of bounded AR (2) model without constant and with dependent errors under different tests of hypothesis have been derived.

Part (2): Simulation Study

- For bounded AR (2) model without constant and with dependent errors the measurement of MSE approve H_a more than the measurement of Thiel's U.

- For bounded AR (2) model without constant and with dependent errors the positive values of ϕ_1 approve H_a more than the negative values of ϕ_1.

- For bounded AR (2) model without constant and with dependent error the generalized inverse G_{71} approve H_a more than the generalized inverses G_{72}, G_{73} and G_{74} in all cases of sample size T, $\bar{c}=-\underline{c}$ and ϕ_1.

- For bounded AR (2) model without constant and with dependent errors and for each sample size T , $\bar{c}=-\underline{c}$ and for generalized inverses G_{72}, G_{73} and G_{74} the values of MSE are decreasing for decreasing of positive values of ϕ_1 and increasing for decreasing of negative values of ϕ_1, while the values of Thiel's U are increasing for both decreasing of positive values of ϕ_1 and decreasing of negative values of ϕ_1 under both the null and alternative hypotheses.

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Unit Root Test of Bounded AR (2) without Constant Model in Case Dependent Errors

اختبار جذر الوحدة لنموذج الانحدار الذاتى المحدود من الرتبة الثانية بدون حد ثابت فى حالة الأخطاء غير المستقلة

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ملخص البحث

فى هذا البحث تم استنتاج اختبار جذر الوحدة لنموذج الانحدار الذاتى المحدود من الرتبة الثانية بدون حد ثابت فى حالة الأخطاء غير المستقلة. واستنتاج التوزيعات التقاربية لمقدرات المربعات الصغرى العادية واحصاءات "ت" فى ظل اختبارت فروض مختلفة. كما تم بناء دراسة محاكاة للمقارنة بين مختلف اختبارات جذر الوحدة وتم استخدام متوسط مربعات الخطأ ومعامل عدم التساوى كمعايير للمقارنة.

الكلمات المفتاحية: نموذج الانحدار الذاتى المحدود من الرتبة الثانية بدون حد ثابت، التوزيعات التقاربية، مقدرات المربعات الصغرى العادية، اختبارات الفروض، احصاءات "ت"، متوسط مربعات الخطأ، معامل عدم التساوى وقوة الاختبار.